Old and New Wavenumber-Explicit Estimates for Boundary Integral Operators in Acoustic Scattering

Simon Chandler-Wilde

Department of Mathematics and Statistics
University of Reading
s.n.chandler-wilde@reading.ac.uk

Joint work with many collaborators, including (present here in Nice) Euan Spence and my PhD student Siavash Sadeghi.

Semiclapp, Nice, May 2024

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This talk is about what I've worked on throughout my career, namely

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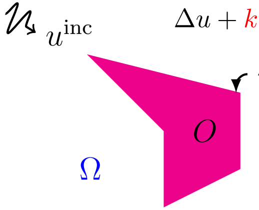
I'll consider two, related, **variants** of this problem.

Variant 1: Scattering Problem



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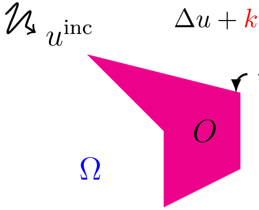

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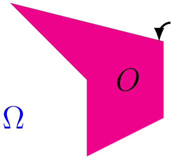
Variant 2: Source Problem (source is f)

$\Delta v + k^2 v = f$ (compactly supported)



$v = 0$

v satisfies SRC, i.e.
$\partial_r v - \mathrm{i} k v = o(r^{-(d-1)/2})$ as $r := |x| \to \infty$.

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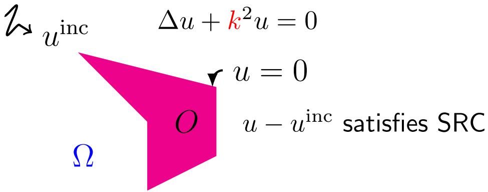
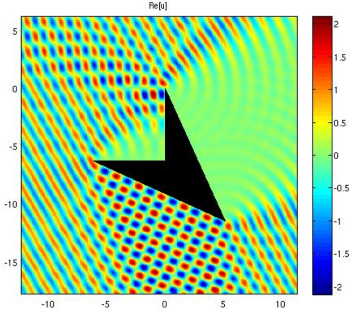


Example 2D Boundary Element Method (BEM) computation when $u^{\mathrm{inc}}(x) = \exp(\mathrm{i}kx \cdot \hat{d})$ is a plane wave and O is a polygon, using an asymptotic-numerical hp-BEM (C-W, Hewett, Langdon, Twigger, 2015) and $O(1)$ degrees of freedom as $k \to \infty$.



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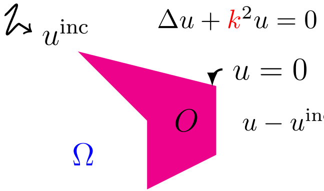
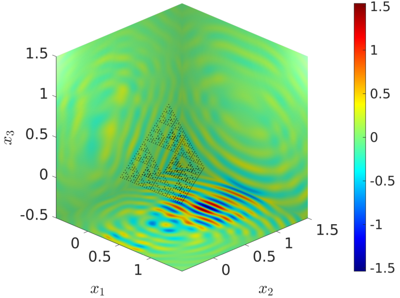


$u - u^{\mathrm{inc}}$ satisfies SRC

Example 3D BEM computation when $u^{\mathrm{inc}}(x) = \exp(\mathrm{i}kx \cdot \hat{d})$ is a plane wave and O is a Sierpinski tetrahedron (Caetano, C-W, Claeys, Gibbs, Hewett, Moiola 2024)



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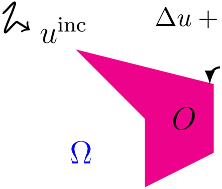

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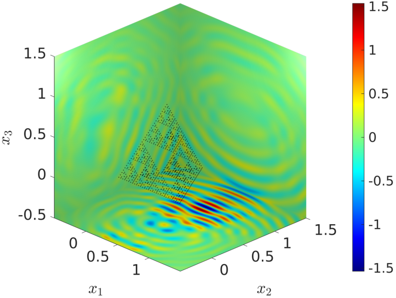

Mostly we will assume O is **Lipschitz or smoother**. At the end we treat **general compact** O.

Our aims

1. To recall old and new reformulations of the above scattering problem as boundary integral equations, each taking the form

$$A_k \phi = g$$

where $\phi \in H$, some Hilbert space of functions on O, $g \in H^*$, and $A_k : H \to H^*$ is some boundary integral operator.

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Our **motivations from numerical analysis** are that bounds on $\|A_k^{-1}\|$, together with bounds on $\|A_k\|$, see, e.g., Han & Tacy (2015), C-W et al (2009, 2020), which give us bounds on the **condition number** $\mathrm{cond}(A_k) := \|A_k\|\|A_k^{-1}\|$:

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- Indicate sensitivity of the numerical solution to uncertainty or discretisation errors
- Lead to bounds on condition numbers at a discrete level (Betcke et al 2011), which are related to the convergence of iterative solvers, e.g. GMRES

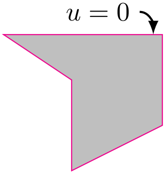
Overview of Talk

1. What is this talk about?
2. A key tool: resolvent estimates
   - What are they?
   - The known estimates and their geometries
3. Applications to Boundary Integral Equations
   - The standard Burton-Miller 2nd kind BIE
   - The standard 1st kind BIE
   - A new 1st kind IE
4. Conclusions

What is a resolvent estimate?

u satisfies SRC

$$\Delta u + k^2 u = f$$



It's a bound, explicit in k, on the (outgoing) cutoff resolvent for this problem, i.e. on

$$\|\chi(-\Delta_D - k^2)^{-1}\chi\|_{L^2 \to L^2},$$

where $\chi \in C_0^\infty$ and $-\Delta_D$ is the **Dirichlet Laplacian**.

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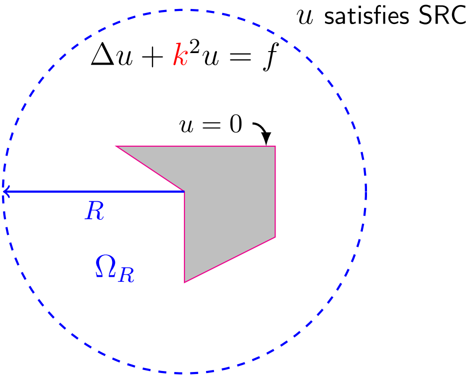

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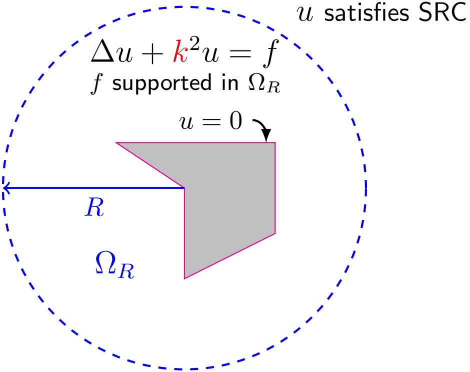

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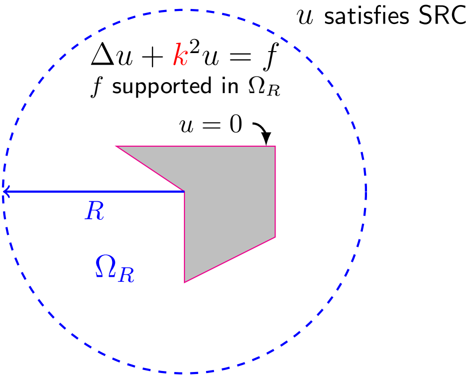
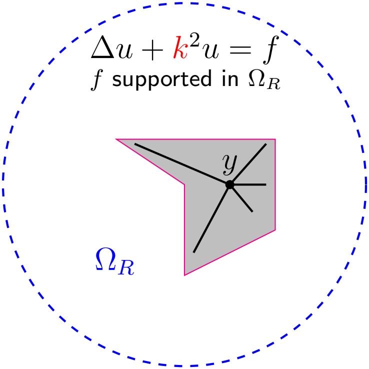


Explicitly, it's the wavenumber-explicit bound that, for all $R, k_0 > 0$ and some specified $c(k)$,

$$\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0.$$

$A \lesssim B$ means $A \leq CB$, where $C > 0$ independent of k and f, but **depends on R and k_0.**

The known estimates and their geometries



Suppose that O is **star-shaped**, i.e., for some $y \in O$,

$$x \in O \Rightarrow sy + (1-s)x \in O, \quad \forall s \in (0,1).$$

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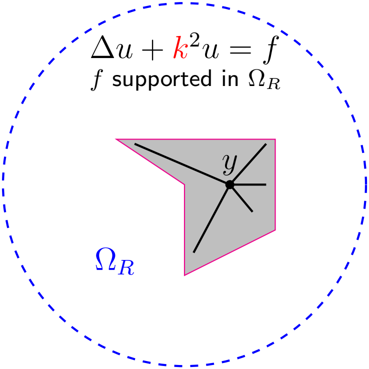

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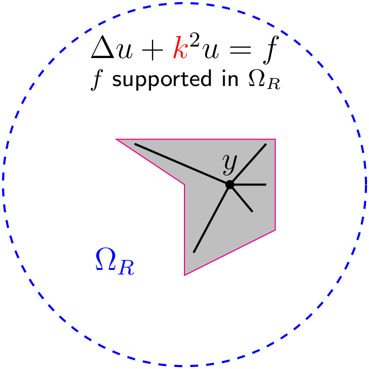

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This is a sharp bound: achieved by $u(x) = \chi(x) \exp(\mathrm{i} k x_1)$, if $\chi \in C_0^\infty(\Omega_R)$.

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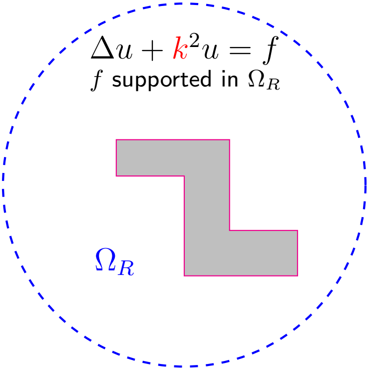

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holds, more generally, for **nontrapping** obstacles (C^∞: Morawetz, Ralston, Strauss 1977, Vainberg 1975, Melrose & Sjöstrand 1982; polygon: Baskin & Wunsch 2013).

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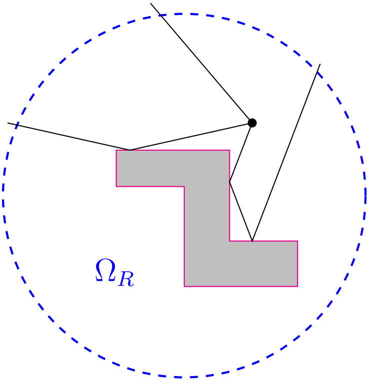

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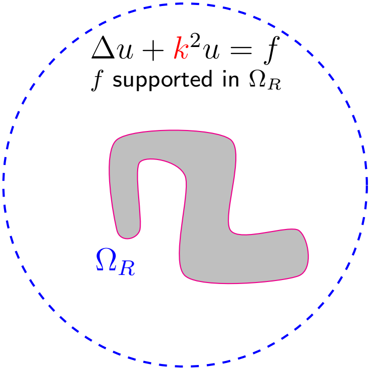
Nontrapping: there exists $T > 0$ such that all the billiard trajectories starting in Ω_R at time zero leave Ω_R by time T.

The known estimates and their geometries



General C^∞ **"worst case"** bound (Burq 1998): for some $\alpha > 0$,

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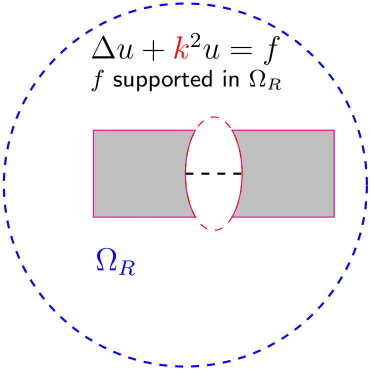

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Achieved for some $k_m \to \infty$ when there is **elliptic, stable trapping** (Cardoso, Popov 2002).

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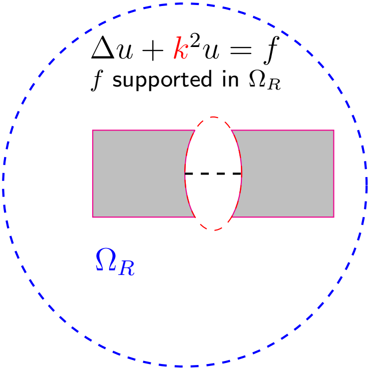

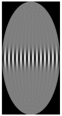

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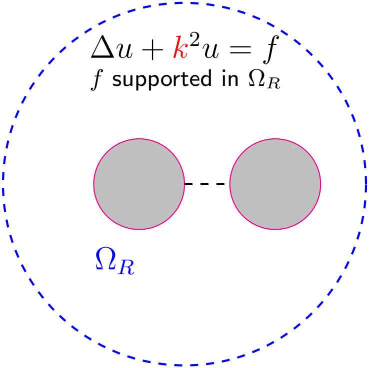
Achieved for some $k_m \to \infty$ when there is **elliptic, stable trapping** (Cardoso, Popov 2002). In the above geometry (Betcke et al 2011) by the **quasimode**

$$u(x) := \chi(x) u^{\text{eig}}_{k_m}(x),$$

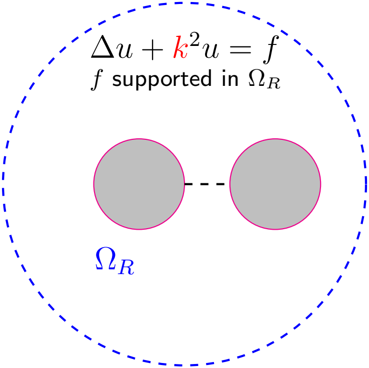
with $\chi \in C_0^\infty(\Omega_R)$ such that $\chi = 1$ near the trapped ray and $u = 0$ on $\partial\Omega$.

The known estimates and their geometries



Two or more C^∞ strictly convex, positive curvature obstacles
(Ikawa 1988, Burq 2004), example of **hyperbolic, unstable trapping**

The known estimates and their geometries

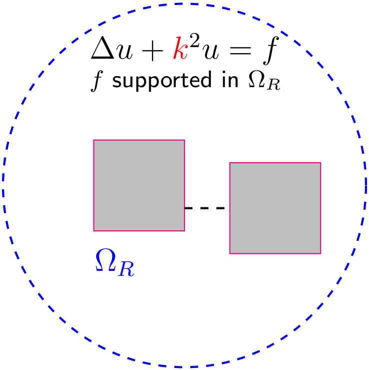


Two or more C^∞ strictly convex, positive curvature obstacles (Ikawa 1988, Burq 2004), example of **hyperbolic, unstable trapping**

$$\|u\|_{L^2(\Omega_R)} \lesssim k^{-1} \log(2+k) \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k^{-1}\log(2+k),$$

so **only logarithmically worse** than the nontrapping case
- cf. Semiclassical Scattering Exercise Session 2!

The known estimates: parabolic, neutral trapping



Theorem (C-W, Spence, Gibbs, Smyshlyaev 2020)

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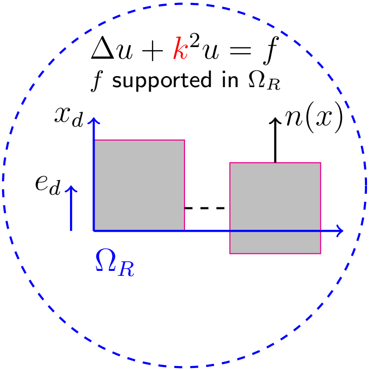

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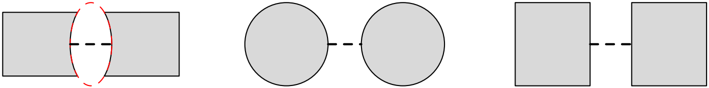
Applies to a general Lipschitz obstacle class, in particular when

$$x_d e_d \cdot n(x) \geq 0 \quad \textit{on the boundary}$$

Recap of resolvent estimates

$$\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0,$$

where $c(k) = k^{-1}$ for **nontrapping** obstacles, and



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elliptic & C^∞

$c(k) = k^{-1}\log(2+k)$
hyperbolic & C^∞

$c(k) = k$
parabolic

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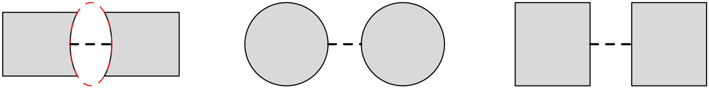

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Additionally (Lafontaine, Spence, Wunsch 2021), if O is Lipschitz and we **avoid the wavenumber sequences for which there is strong trapping** then (almost) $c(k) = k^{5d/2}$

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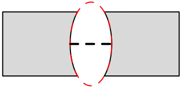

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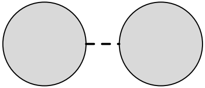

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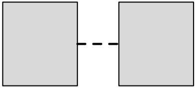

$c(k) = k$
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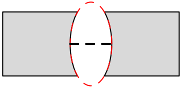
Additionally (Lafontaine, Spence, Wunsch 2021), if O is Lipschitz and we **avoid the wavenumber sequences for which there is strong trapping** then (almost) $c(k) = k^{5d/2}$: precisely, given $\delta, \varepsilon > 0$ there exists $E \subset [k_0, \infty)$ with $|E| < \varepsilon$ such that the resolvent estimate holds with

$$c(k) = k^{5d/2+\delta}, \quad \text{for} \quad k \in [k_0, \infty) \setminus E.$$

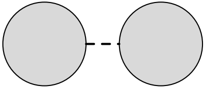
Recap of resolvent estimates

$$\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0,$$

where $c(k) = k^{-1}$ for **nontrapping** obstacles, and



$c(k) = \exp(\alpha k)$
elliptic & C^∞



$c(k) = k^{-1}\log(2 + k)$
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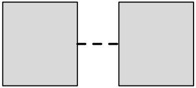

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See Siavash Sadeghi's poster for more info ...

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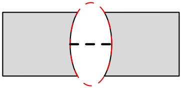

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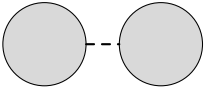

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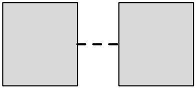

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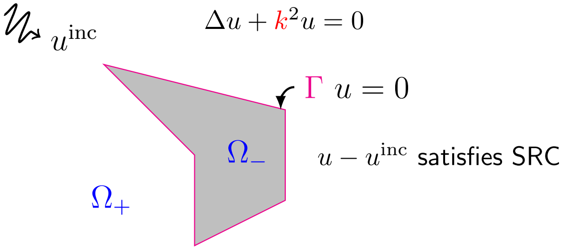
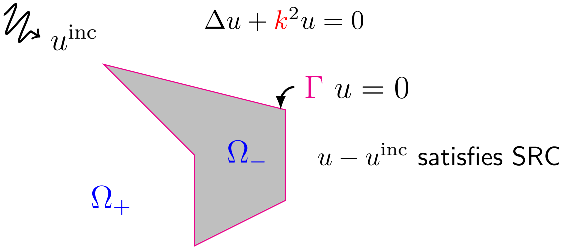
Overview of Talk



Let $\Omega_+ := \Omega$ and assume $\Omega_- := \text{int}(O)$ is Lipschitz and $O = \overline{\Omega_-}$, and put $\Gamma := \partial O = \partial \Omega_\pm$.

Integral Equations and k-Explicit Bounds



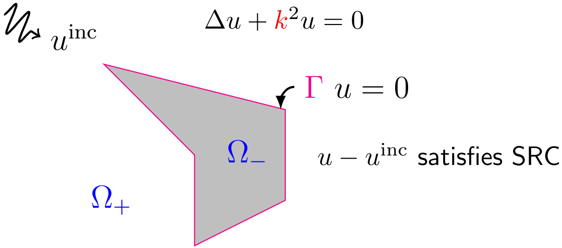
Let $\Omega_+ := \Omega$ and assume $\Omega_- := \mathrm{int}(O)$ is Lipschitz and $O = \overline{\Omega_-}$, and put $\Gamma := \partial O = \partial \Omega_\pm$.

Theorem (Green's Representation Theorem)

$$u(x) = u^{\mathrm{inc}}(x) - \int_\Gamma \Phi(x,y)\partial_n^+ u(y)\, ds(y), \quad x \in \Omega_+,$$

where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \quad := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$



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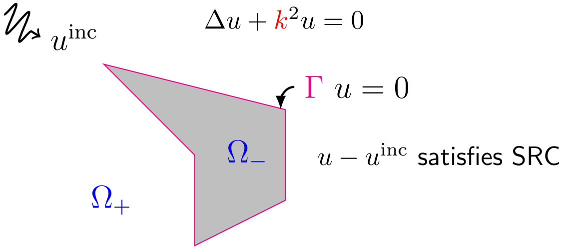


Theorem (Green's Representation Theorem)

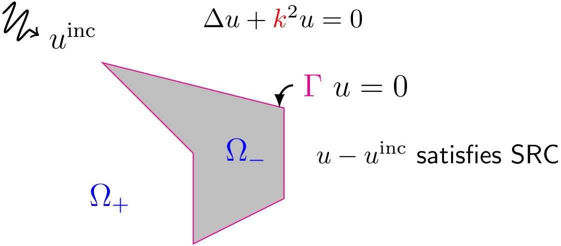
$$u(x) = u^{\mathrm{inc}}(x) - \int_\Gamma \Phi(x,y)\partial_n^+ u(y)\, ds(y), \quad x \in \Omega_+.$$

Taking a linear combination of Dirichlet (γ_+) and Neumann (∂_n^+) traces, we get the **boundary integral equation** (Burton & Miller 1971)

$$\frac{1}{2}\partial_n^+ u(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} - \mathrm{i}\eta \Phi(x,y) \right) \partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma,$$

where

$$F := \partial_n^+ u^{\mathrm{inc}} - \mathrm{i}\eta\gamma_+ u^{\mathrm{inc}}.$$



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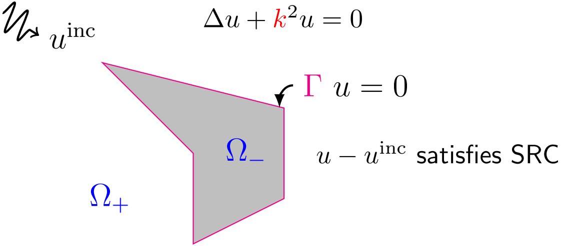
in operator form

$$A_{k,\eta} \partial_n^+ u = F := \partial_n^+ u^{\mathrm{inc}} - \mathrm{i}\eta \gamma_+ u^{\mathrm{inc}}.$$



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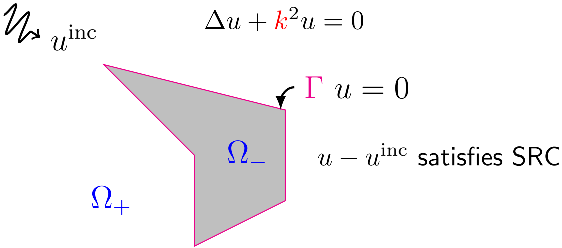
Theorem (Burton & Miller 1971, Mitrea 1996, C-W & Langdon 2007)

If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.



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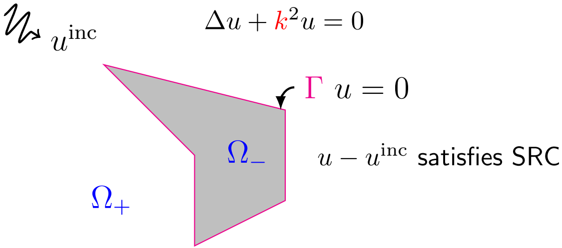
If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.

The standard choice is $\eta = k$, and with this choice we have

$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$$

if Ω_- is **star-shaped** (C-W, Monk 2008) or C^∞ and **nontrapping** (Baskin, Spence, Wunsch 2016).



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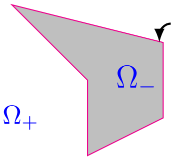
if Ω_- is **star-shaped** (C-W, Monk 2008) or C^∞ and **nontrapping** (Baskin, Spence, Wunsch 2016). Where does this bound come from and what if Ω_- is **trapping**?

$\Delta u + k^2 u = f \in L^2(\Omega_+)$, compactly supported



$\Gamma \; u = g \in H^1(\Gamma)$

u satisfies SRC

A recipe for bounding $\|A_{k,k}^{-1}\|$ (Baskin, Spence, Wunsch 2016, C-W, Spence, Gibbs, Smyshlyaev 2020)

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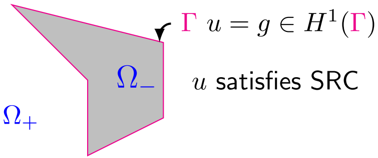

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Step 1 (Resolvent Estimate). Show that, for every $R > 0$, if $g = 0$,

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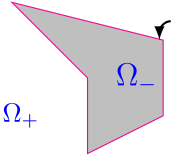

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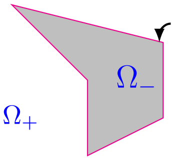
Step 2 (DtN Map Bound). It follows that, if $f = 0$,

$$\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim k c(k) \left(\|\nabla_\Gamma g\|_{L^2(\Gamma)} + k\|g\|_{L^2(\Gamma)}\right)$$

$\Delta u + k^2 u = f \in L^2(\Omega_+)$, compactly supported



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$$A_{k,k}^{-1} = I - (DtN_k^+ - \mathrm{i}k) ItD_k^-$$

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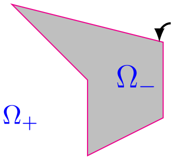

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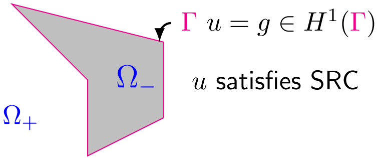

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for general Lipschitz Ω_-.

Recap of resolvent estimates

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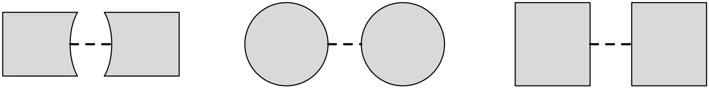

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Further, for all Lipschitz O and all $\delta, \varepsilon > 0$, there exists E with $|E| \leq \varepsilon$, such that

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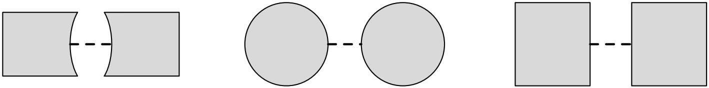

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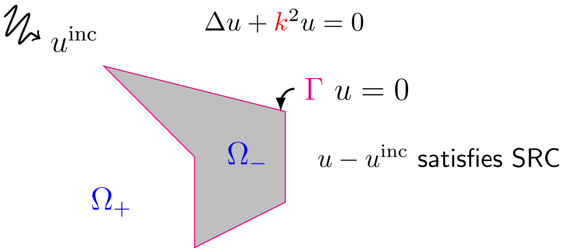
Applying our general recipe

$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{3/2}c(k),$$

in all these cases, indeed $\|A_{k,k}^{-1}\| \lesssim kc(k)$ if O is C^∞.

1st Kind BIE (see Siavash Sadeghi's poster for details)



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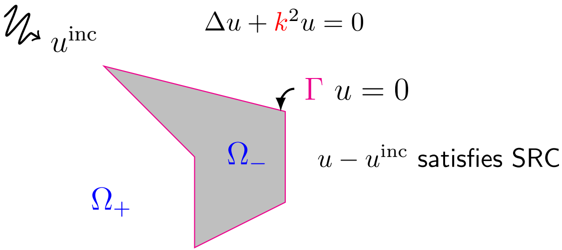
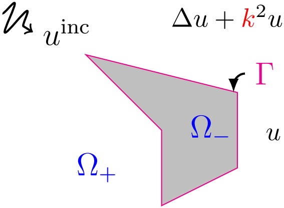


Theorem (Green's Representation Theorem)

$$u(x) = u^{\mathrm{inc}}(x) - \int_\Gamma \Phi(x,y)\partial_n^+ u(y)\, ds(y), \quad x \in \Omega_+.$$

Taking the Dirichlet (γ_+) trace we get the **1st kind boundary integral equation**

$$\int_\Gamma \Phi(x,y)\partial_n^+ u(y) ds(y) = F(x) := \gamma_+ u^{\mathrm{inc}}(x), \quad x \in \Gamma,$$



$\Delta u + k^2 u = 0$

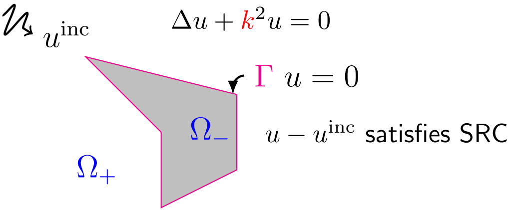
$u = 0$

$u - u^{\text{inc}}$ satisfies SRC

$$\int_\Gamma \Phi(x,y)\partial_n^+ u(y)ds(y) = F(x), \quad x \in \Gamma,$$

in operator form

$$S_k \partial_n^+ u = F := \gamma_+ u^{\text{inc}}.$$



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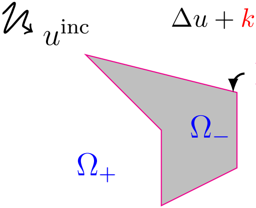
It is standard that S_k invertible iff $k^2 \notin \text{spec}(-\Delta_D(\Omega_-))$



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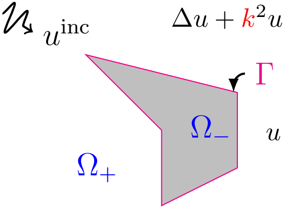
$$S_k^{-1} = DtN_k^- - DtN_k^+.$$



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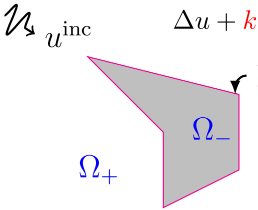
Further, by our previous recipe, $\|DtN_k^+\| \lesssim k c(k)$.



$\Delta u + k^2 u = 0$

$\Gamma \quad u = 0$

$u - u^{\rm inc}$ satisfies SRC

$$\int_\Gamma \Phi(x,y)\partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma,$$

in operator form

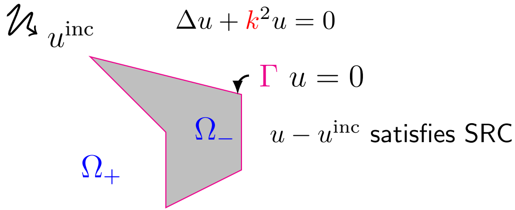
$$S_k \partial_n^+ u = F := \gamma_+ u^{\rm inc}.$$

It is standard that S_k invertible iff $k^2 \notin \mathrm{spec}(-\Delta_D(\Omega_-))$, indeed

$$S_k^{-1} = DtN_k^- - DtN_k^+.$$

Further, by our previous recipe, $\|DtN_k^+\| \lesssim kc(k)$. Similarly (C-W & Sadeghi 2024+),

$$\|DtN_k^-\| \lesssim kc_-(k)$$



$$\int_\Gamma \Phi(x,y)\partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma,$$

in operator form

$$S_k \partial_n^+ u = F := \gamma_+ u^{\text{inc}}.$$

It is standard that S_k invertible iff $k^2 \notin \text{spec}(-\Delta_D(\Omega_-))$, indeed

$$S_k^{-1} = DtN_k^- - DtN_k^+.$$

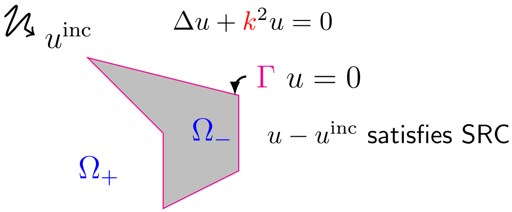
Further, by our previous recipe, $\|DtN_k^+\| \lesssim kc(k)$. Similarly (C-W & Sadeghi 2024+),

$$\|DtN_k^-\| \lesssim kc_-(k) \lesssim \frac{k}{\text{dist}(k^2, \text{spec}(-\Delta_D(\Omega_-)))},$$

using that $-\Delta_D(\Omega_-)$ is self-adjoint, so that

$$c_-(k) = \|(\Delta_D + k^2)^{-1}\|_{L^2(\Omega_-)\to L^2(\Omega_-)} = \left[\text{dist}(k^2, \text{spec}(-\Delta_D(\Omega_-)))\right]^{-1}.$$

Comparing 1st & 2nd Kind BIEs: general Lipschitz Ω_-



2nd kind BIE	**1st kind BIE**
$A_{k,k}\partial_n^+ u = f$ Invertible for all $k > 0$	$S_k \partial_n^+ u = f$ Invertible for $k^2 \notin \text{spec}(-\Delta_D(\Omega_-))$

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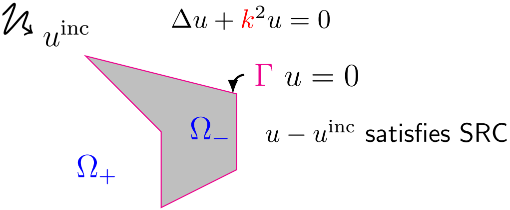

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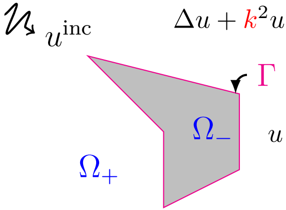
Invertible for all $k > 0$	Invertible for $k^2 \notin \mathrm{spec}(-\Delta_D(\Omega_-))$
$A_{k,k}^{-1} = I - (DtN_k^+ - \mathrm{i}k)ItD_k^-$	$S_k^{-1} = DtN_k^- - DtN_k^+$

Comparing 1st & 2nd Kind BIEs: general Lipschitz Ω_-



$\Delta u + k^2 u = 0$

Γ $u = 0$

$u - u^{\mathrm{inc}}$ satisfies SRC

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$\\|A_{k,k}^{-1}\\| \lesssim k^{(5d+3)/2+\delta}$	$\\|S_k^{-1}\\| \lesssim k^{(5d+2)/2+\delta}$

The bounds in the last row hold for $k \in [k_0, \infty) \setminus E$, with $|E|$ and $\delta > 0$ arbitrarily small; for the S_k^{-1} bound see Siavash's poster.

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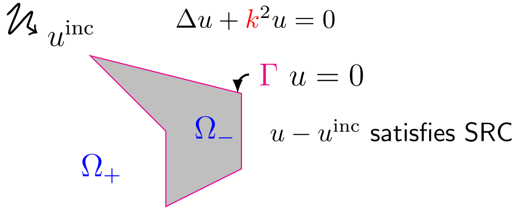

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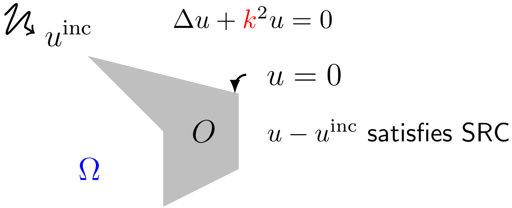
Conjecture.

$$\|DtN_k^+\|,\ \|A_{k,k}^{-1}\|,\ \|S_k^{-1}\| \lesssim k^d \log^2(2+k), \quad k \in [k_0, \infty) \setminus E.$$

Overview of Talk

1st kind IE for general compact O (Caetano et al 2024)



Recall O is compact and $\Omega := \mathbb{R}^d \setminus O$ is connected, and assume $u^{\mathrm{inc}} \in H^{1,\mathrm{loc}}(\mathbb{R}^d)$.

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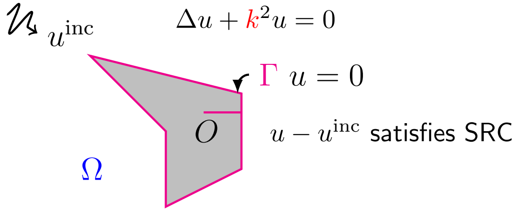

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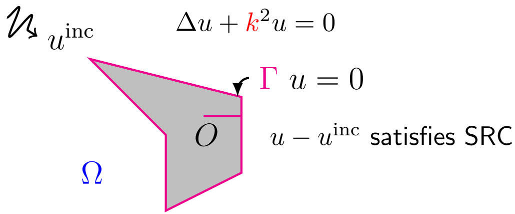

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$$\mathcal{A}\psi(x) := \int_{\mathbb{R}^d} \Phi(x,y)\psi(y)\,\mathrm{d}y, \quad \text{for} \quad \psi \in L^2_{\mathrm{comp}}(\mathbb{R}^d), \ \ x \in \mathbb{R}^d.$$

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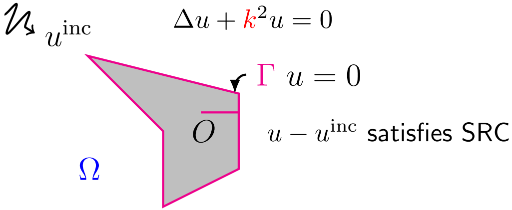

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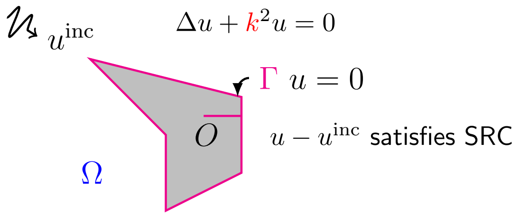

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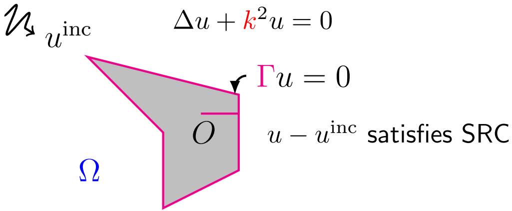
$$\chi u \in \widetilde{H}^1(\Omega^*) := \overline{C_0^\infty(\Omega^*)}^{H^1(\mathbb{R}^d)}.$$



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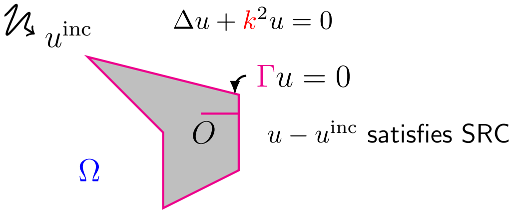

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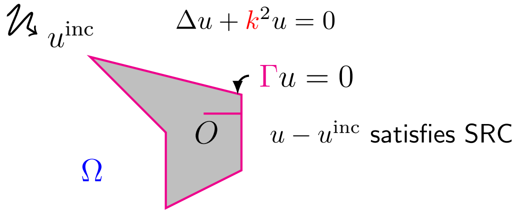
$$P(\chi u) = 0$$



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$$P(\chi u) = 0 \quad \Leftrightarrow \quad \mathbf{S}_k\phi = g := -P(\chi u^{\rm inc}), \quad \text{where} \quad \mathbf{S}_k\phi := P(\chi\mathcal{A}\phi).$$

$\mathbf{S}_k\phi = g$ can be solved by Galerkin BEM (Caetano et al 2024)

$\mathbf{S}_k \phi = g$ can be solved by Galerkin BEM (Caetano et al 2024)

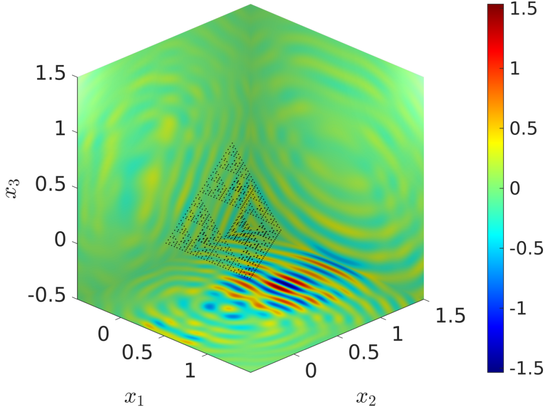
Example computation: $\Gamma = O = \partial O =$ Sierpinski tetrahedron,

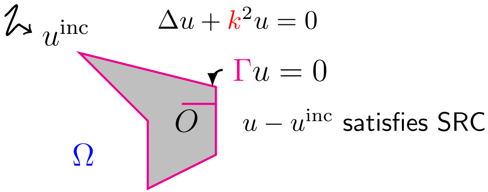
$$u = u^{\mathrm{inc}} + \mathcal{A}\phi,$$

where $\phi \in H_\Gamma^{-1} := \{\psi \in H^{-1}(\mathbb{R}^d) : \mathrm{supp}(\psi) \subset \Gamma\}$ satisfies

$$\mathbf{S}_k \phi = g := -P(\chi u^{\mathrm{inc}}), \quad \text{where} \quad \mathbf{S}_k \phi := P(\chi \mathcal{A}\phi).$$

Plotted is the **scattered field** $\mathcal{A}\phi$.





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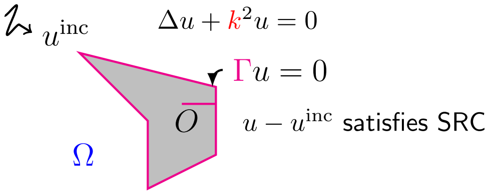

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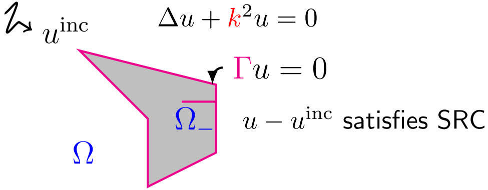

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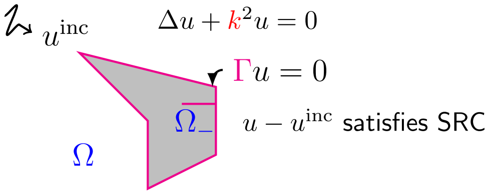
Theorem (Caetano et al 2024, C-W & Sadeghi 2024+)

Let $\Omega_- := O \setminus \Gamma$, *and* $c(k)$ *and* $c_-(k)$ *denote the bounds in the resolvent estimates for* Ω *and* Ω_-. *Then* $\mathbf{S}_k : H^{-1}_\Gamma \to (H^{-1}_\Gamma)^*$ *is invertible iff* $k^2 \notin \mathrm{spec}(-\Delta_D(\Omega_-))$, *and*

$$\|\mathbf{S}_k^{-1}\| \lesssim k^2 c(k) + k^2 c_-(k)$$



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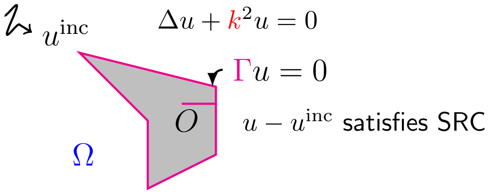

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Conjecture

Let $\Omega_- := O \setminus \Gamma$, and $c(k)$ and $c_-(k)$ denote the bounds in the resolvent estimates for Ω and Ω_-. Then

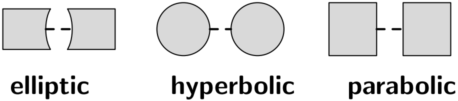
$$\|\mathbf{S}_k^{-1}\| \lesssim k c(k) + k c_-(k) \lesssim k^{d+1}\log^2(2+k),$$

for $k \in [k_0,\infty) \setminus E$ with $|E| \le \varepsilon$, **for every** *obstacle O.*

Conclusions

In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles, including



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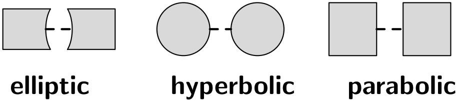

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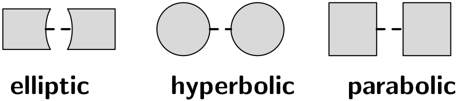

- The standard 1st and 2nd kind BIEs when O is Lipschitz, and a new 1st kind integral equation for general compact O
- How resolvent estimates lead in a "black box" way to:
  - bounds on (exterior and interior) DtN maps
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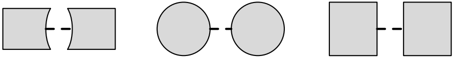

elliptic **hyperbolic** **parabolic**

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Lots of open problems:

- our conjectures above;
- are all our estimates for integral operators in terms of resolvent estimates sharp?

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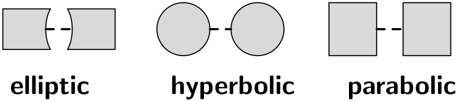
In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles, including

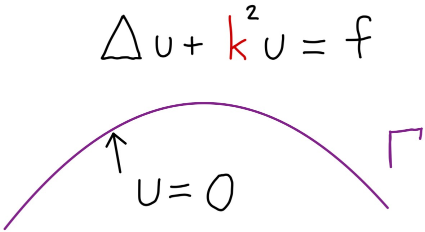


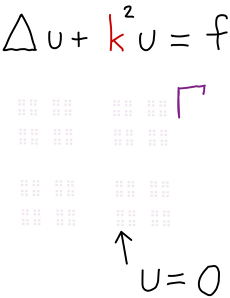
- The standard 1st and 2nd kind BIEs when O is Lipschitz, and a new 1st kind integral equation for general compact O
- How resolvent estimates lead in a "black box" way to:
    - bounds on (exterior and interior) DtN maps
    - bounds on $\|A_{k,k}^{-1}\|$, $\|S_k^{-1}\|$, $\|\mathbf{S}_k^{-1}\|$

Lots of open problems:

- our conjectures above;
- are all our estimates for integral operators in terms of resolvent estimates sharp?
- resolvent estimates are missing, or need sharpening, for many configurations, notably where the obstacle is non-smooth, e.g. Lipschitz or fractal

Example open problems





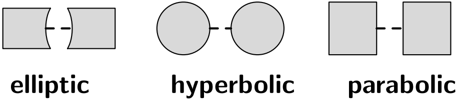
Thin curved screen: resolvent estimate? Bound on $\|S_k^{-1}\|$?

Fractal: resolvent estimate? Sharp bound on $\|\mathbf{S}_k^{-1}\|$?

Conclusions

In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles, including



- The Morawetz/Rellich identity method for proving these estimates
- The standard 1st and 2nd kind BIEs when O is Lipschitz, and a new 1st kind integral equation for general compact O
- How resolvent estimates lead in a "black box" way to:
  - bounds on (exterior and interior) DtN maps
  - bounds on $\|A_{k,k}^{-1}\|$, $\|S_k^{-1}\|$, $\|\mathbf{S}_k^{-1}\|$

Lots of open problems:

- our conjectures above;
- are all our estimates for integral operators in terms of resolvent estimates sharp?
- resolvent estimates are missing, or need sharpening, for many configurations, notably where the obstacle is non-smooth, e.g. Lipschitz or fractal

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