Old and New Wavenumber-Explicit Estimates for Boundary Integral Operators in Acoustic Scattering

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Joint work with many collaborators, including (present here in Nice) Euan Spence and my PhD student Siavash Sadeghi.

Semiclapp, Nice, May 2024
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This talk is about what I’ve worked on throughout my career, namely

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Focus today on scattering in \( \mathbb{R}^d \) \((d \geq 2)\) by compact obstacle, \( O \), with Dirichlet boundary conditions, the so-called sound-soft case in acoustic terminology.
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Focus today on \textbf{scattering} in \( \mathbb{R}^d (d \geq 2) \) by \textbf{compact obstacle}, \( O \), with \textbf{Dirichlet} boundary conditions, the so-called \textbf{sound-soft} case in acoustic terminology.

The wave propagation is in \( \Omega := \mathbb{R}^d \setminus O \), the complement of and exterior of \( O \), which we assume is \textbf{connected}. 
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Focus today on **scattering** in \( \mathbb{R}^d \) \((d \geq 2)\) by **compact obstacle**, \( O \), with **Dirichlet** boundary conditions, the so-called **sound-soft** case in acoustic terminology.

The wave propagation is in \( \Omega := \mathbb{R}^d \setminus O \), the complement of and exterior of \( O \), which we assume is **connected**.

I’ll consider two, related, **variants** of this problem.
Variant 1: Scattering Problem

\[ \nabla \cdot u^{\text{inc}} = 0 \]
\[ \Delta u + k^2 u = 0 \]
\[ u = 0 \]
\[ u - u^{\text{inc}} \text{ satisfies Sommerfeld rad. cond. (SRC)} \]
Variant 1: Scattering Problem

\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \quad u - u^{\text{inc}} \text{ satisfies Sommerfeld rad. cond. (SRC)} \]

Variant 2: Source Problem (source is \( f \))

\[ \Delta v + k^2 v = f \text{ (compactly supported)} \]

\[ v = 0 \quad v \text{ satisfies SRC, i.e.} \]

\[ \partial_r v - i k v = o(r^{-(d-1)/2}) \text{ as } r := |x| \to \infty. \]
Variant 1: Scattering Problem

\[ \nabla^2 u + k^2 u = 0 \]

\[ u = 0 \quad \text{on} \quad \Omega \]

\[ u - u^{\text{inc}} \] satisfies SRC

Example 2D Boundary Element Method (BEM) computation when \( u^{\text{inc}}(x) = \exp(ikx \cdot \hat{d}) \) is a plane wave and \( \Omega \) is a polygon, using an asymptotic-numerical \( hp \)-BEM (C-W, Hewett, Langdon, Twigger, 2015) and \( O(1) \) degrees of freedom as \( k \to \infty \).
Variant 1: Scattering Problem

\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \]

\[ u - u^{\text{inc}} \text{ satisfies SRC} \]

\( \Omega \)

Example 3D BEM computation when \( u^{\text{inc}}(x) = \exp(ikx \cdot \hat{d}) \) is a plane wave and \( \Omega \) is a Sierpinski tetrahedron (Caetano, C-W, Claeys, Gibbs, Hewett, Moiola 2024)
Variant 1: Scattering Problem

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Example 3D BEM computation when \( u^{\text{inc}}(x) = \exp(ikx \cdot \hat{d}) \) is a plane wave and \( \Omega \) is a Sierpinski tetrahedron (Caetano, C-W, Claeys, Gibbs, Hewett, Moiola 2024)

Mostly we will assume \( \Omega \) is Lipschitz or smoother. At the end we treat general compact \( \Omega \).
Our aims

1. To recall old and new reformulations of the above scattering problem as boundary integral equations, each taking the form

\[ A_k \phi = g \]

where \( \phi \in H \), some Hilbert space of functions on \( O \), \( g \in H^* \), and \( A_k : H \to H^* \) is some boundary integral operator.

2. To obtain bounds on \( \| A_k^{-1} \| \) that are explicit in \( k \), and that make clear the influence of the geometry of \( O \)- and the usefulness of resolvent estimates!

Our motivations from numerical analysis are that bounds on \( \| A_k^{-1} \| \), together with bounds on \( \| A_k \| \), see, e.g., Han & Tacy (2015), C-W et al (2009, 2020), which give us bounds on the condition number \( \text{cond}(A_k) := \| A_k \| \| A_k^{-1} \| \): Are needed for wavenumber-explicit bounds on errors in BEM, e.g., \( hp \)-Galerkin BEM (Löndorf & Melenk 2011) Indicate sensitivity of the numerical solution to uncertainty or discretisation errors Lead to bounds on condition numbers at a discrete level (Betcke et al 2011), which are related to the convergence of iterative solvers, e.g. GMRES
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- Lead to bounds on condition numbers at a discrete level (Betcke et al 2011), which are related to the convergence of iterative solvers, e.g. GMRES
Overview of Talk

1. What is this talk about?

2. A key tool: resolvent estimates
   - What are they?
   - The known estimates and their geometries

3. Applications to Boundary Integral Equations
   - The standard Burton-Miller 2nd kind BIE
   - The standard 1st kind BIE
   - A new 1st kind IE

4. Conclusions
What is a resolvent estimate?

\[ \Delta u + k^2 u = f \]

It’s a bound, explicit in \( k \), on the (outgoing) cutoff resolvent for this problem, i.e. on

\[ \| \chi (-\Delta_D - k^2)^{-1} \chi \|_{L^2 \to L^2}, \]

where \( \chi \in C_0^\infty \) and \( -\Delta_D \) is the **Dirichlet Laplacian**.
What is a resolvent estimate?

\[ \Delta u + k^2 u = f \]

\[ u = 0 \]

\[ \Omega_R \]

\[ u \text{ satisfies SRC} \]
What is a resolvent estimate?

Explicitly, it's the wavenumber-explicit bound that, for all \( R, k_0 > 0 \) and some specified \( c(k) \),
\[
\|u\|_{L^2(\Omega_R)} \lesssim c(k) \|f\|_{L^2(\Omega_R)},
\]
for \( k \geq k_0 > 0 \).

\( A \lesssim B \) means \( A \leq C B \), where \( C > 0 \) independent of \( k \) and \( f \), but depends on \( R \) and \( k_0 \).
What is a resolvent estimate?

 Explicitly, it’s the wavenumber-explicit bound that, for all $R, k_0 > 0$ and some specified $c(k)$,

$$\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0.$$

$A \lesssim B$ means $A \leq CB$, where $C > 0$ independent of $k$ and $f$, but depends on $R$ and $k_0$. 
The known estimates and their geometries

Suppose that $O$ is **star-shaped**, i.e., for some $y \in O$,

$$x \in O \Rightarrow sy + (1 - s)x \in O, \quad \forall s \in (0, 1).$$
The known estimates and their geometries

\[ \Delta u + k^2 u = f \]
\[ f \text{ supported in } \Omega_R \]

Suppose that \( O \) is star-shaped, i.e., for some \( y \in O \),
\[ x \in O \Rightarrow sy + (1 - s)x \in O, \quad \forall s \in (0, 1). \]

Then (Morawetz 1975, C-W & Monk 2008)
\[ \|u\|_{L^2(\Omega_R)} \lesssim k^{-1}\|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k^{-1}. \]
The known estimates and their geometries

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\[ \|u\|_{L^2(\Omega_R)} \lesssim k^{-1} \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k^{-1}. \]

This is a sharp bound: achieved by \( u(x) = \chi(x) \exp(ikx_1) \), if \( \chi \in C_0^\infty(\Omega_R) \).
The known estimates and their geometries

\[ \Delta u + k^2 u = f \]
\[ f \text{ supported in } \Omega_R \]

The same bound
\[ \| u \|_{L^2(\Omega_R)} \lesssim k^{-1} \| f \|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k^{-1}, \]
holds, more generally, for nontrapping obstacles (\( C^\infty \): Morawetz, Ralston, Strauss 1977, Vainberg 1975, Melrose & Sjöstrand 1982; polygon: Baskin & Wunsch 2013).
The known estimates and their geometries

The same bound

\[ \|u\|_{L^2(\Omega_R)} \lesssim k^{-1} \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k^{-1}, \]


**Nontrapping**: there exists \( T > 0 \) such that all the billiard trajectories starting in \( \Omega_R \) at time zero leave \( \Omega_R \) by time \( T \).
The known estimates and their geometries

\[ \Delta u + k^2 u = f \]
\[ f \text{ supported in } \Omega_R \]

General $C^\infty$ "worst case" bound (Burq 1998): for some $\alpha > 0$,

\[ \|u\|_{L^2(\Omega_R)} \lesssim \exp(\alpha k)\|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = \exp(\alpha k). \]
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Achieved for some $k_m \to \infty$ when there is elliptic, stable trapping (Cardoso, Popov 2002).
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Achieved for some \( k_m \to \infty \) when there is elliptic, stable trapping (Cardoso, Popov 2002). In the above geometry (Betcke et al 2011) by the quasimode

\[ u(x) := \chi(x) u_{k_m}^{\text{eig}}(x), \]

with \( \chi \in C^\infty_0(\Omega_R) \) such that \( \chi = 1 \) near the trapped ray and \( u = 0 \) on \( \partial \Omega \).
The known estimates and their geometries

\[ \Delta u + k^2 u = f \]

\( f \) supported in \( \Omega_R \)

Two or more \( C^\infty \) strictly convex, positive curvature obstacles (Ikawa 1988, Burq 2004), example of hyperbolic, unstable trapping
The known estimates and their geometries

\[ \Delta u + k^2 u = f \]
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Two or more \( C^\infty \) strictly convex, positive curvature obstacles (Ikawa 1988, Burq 2004), example of hyperbolic, unstable trapping

\[ \| u \|_{L^2(\Omega_R)} \lesssim k^{-1} \log(2 + k) \| f \|_{L^2(\Omega_R)}, \]  
\[ \text{i.e. } c(k) = k^{-1} \log(2 + k), \]

so only logarithmically worse than the nontrapping case
- cf. Semiclassical Scattering Exercise Session 2!
The known estimates: parabolic, neutral trapping

\[ \Delta u + k^2 u = f \]
\[ f \text{ supported in } \Omega_R \]

\[ \|u\|_{L^2(\Omega_R)} \lesssim k \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k. \]
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\[ \|u\|_{L^2(\Omega_R)} \lesssim k \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = k. \]

**Theorem (C-W, Spence, Gibbs, Smyshlyaev 2020)**

Applies to a general Lipschitz obstacle class, in particular when

\[ x_d e_d \cdot n(x) \geq 0 \quad \text{on the boundary} \]
Recap of resolvent estimates

\[ \|u\|_{L^2(\Omega_R)} \lesssim c(k) \|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0, \]

where \( c(k) = k^{-1} \) for nontrapping obstacles, and

- elliptic & \( C^{\infty} \):
  \[ c(k) = \exp(\alpha k) \]
- hyperbolic & \( C^{\infty} \):
  \[ c(k) = k^{-1} \log(2 + k) \]
- parabolic:
  \[ c(k) = k \]

Additionally (Lafontaine, Spence, Wunsch 2021), if \( \Omega \) is Lipschitz and we avoid the wavenumber sequences for which there is strong trapping then (almost)

\[ c(k) = k^{\frac{d}{2}} \pm \delta \]

precisely, given \( \delta, \varepsilon > 0 \) there exists \( E \subset [k_0, \infty) \) with \( |E| < \varepsilon \) such that the resolvent estimate holds with \( c(k) = k^{\frac{d}{2}} + \delta \), for \( k \in [k_0, \infty) \setminus E \).

See Siavash Sadeghi's poster for more info... or talk to David, Euan or Jared!
Recap of resolvent estimates

\[ \|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0, \]

where \( c(k) = k^{-1} \) for nontrapping obstacles, and

- \( c(k) = \exp(\alpha k) \) \text{ elliptic & } C^\infty
- \( c(k) = k^{-1} \log(2 + k) \) \text{ hyperbolic & } C^\infty
- \( c(k) = k \) \text{ parabolic}

Additionally (Lafontaine, Spence, Wunsch 2021), if \( O \) is Lipschitz and we avoid the wavenumber sequences for which there is strong trapping then (almost) \( c(k) = k^{5d/2} \).
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c(k) = \exp(\alpha k) \quad \text{elliptic & } C^\infty
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\[
c(k) = k \quad \text{parabolic}
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Additionally (Lafontaine, Spence, Wunsch 2021), if \( O \) is Lipschitz and we avoid the wavenumber sequences for which there is strong trapping then (almost) \( c(k) = k^{5d/2} \): precisely, given \( \delta, \varepsilon > 0 \) there exists \( E \subset [k_0, \infty) \) with \( |E| < \varepsilon \) such that the resolvent estimate holds with

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\[ c(k) = \exp(\alpha k) \quad \text{elliptic \& } C^\infty \quad \text{hyperbolic \& } C^\infty \quad \text{parabolic} \]

Additionally (Lafontaine, Spence, Wunsch 2021), if \( O \) is Lipschitz and we avoid the wavenumber sequences for which there is strong trapping then (almost) \( c(k) = k^{5d/2} \): precisely, given \( \delta, \varepsilon > 0 \) there exists \( E \subset [k_0, \infty) \) with \( |E| < \varepsilon \) such that the resolvent estimate holds with

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Recap of resolvent estimates

\[ \|u\|_{L^2(\Omega_R)} \lesssim c(k) \|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0, \]

where \( c(k) = k^{-1} \) for **nontrapping** obstacles, and

\[ c(k) = \exp(\alpha k), \quad c(k) = k^{-1} \log(2 + k), \quad c(k) = k \]

**elliptic & \( C^\infty \)** \hspace{1cm} **hyperbolic & \( C^\infty \)** \hspace{1cm} **parabolic**

Additionally (Lafontaine, Spence, Wunsch 2021), if \( O \) is Lipschitz and we **avoid the wavenumber sequences for which there is strong trapping** then (almost)

\[ c(k) = k^{5d/2} : \text{precisely, given } \delta, \varepsilon > 0 \text{ there exists } E \subset [k_0, \infty) \text{ with } |E| < \varepsilon \text{ such that the resolvent estimate holds with} \]

\[ c(k) = k^{5d/2+\delta}, \quad \text{for } k \in [k_0, \infty) \setminus E. \]

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   - The standard 1st kind BIE
   - A new 1st kind IE

4. Conclusions
Integral Equations and $k$-Explicit Bounds

\[ \nabla u^{\text{inc}} \quad \Delta u + k^2 u = 0 \]

\[ \Gamma u = 0 \]

\[ u - u^{\text{inc}} \text{ satisfies SRC} \]

Let $\Omega_+ := \Omega$ and assume $\Omega_- := \text{int}(O)$ is Lipschitz and $O = \overline{\Omega_-}$, and put $\Gamma := \partial O = \partial \Omega_{\pm}$. 

Theorem (Green's Representation Theorem)

\[ u(x) = u^{\text{inc}}(x) - \int_{\Gamma} \Phi(x, y) \frac{\partial}{\partial n} u(y) \, ds(y), \quad x \in \Omega^+ \]

where

\[ \Phi(x, y) := \frac{i}{4} H_0(1)(k|x - y|) \quad \text{(2D)} \]

\[ := \frac{1}{4\pi} e^{i k |x - y|} |x - y| \quad \text{(3D)} \]
Integral Equations and $k$-Explicit Bounds

Let $\Omega_+ := \Omega$ and assume $\Omega_- := \text{int}(O)$ is Lipschitz and $O = \overline{\Omega_-}$, and put $\Gamma := \partial O = \partial \Omega_\pm$.

**Theorem (Green’s Representation Theorem)**

$$u(x) = u^{\text{inc}}(x) - \int_\Gamma \Phi(x, y) \partial_n^+ u(y) \, ds(y), \quad x \in \Omega_+,$$

where

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \quad (2D), \quad := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x - y|} \quad (3D).$$
Theorem (Green’s Representation Theorem)

\[ u(x) = u^{\text{inc}}(x) - \int_{\Gamma} \Phi(x, y) \partial_{n}^{+} u(y) \, ds(y), \quad x \in \Omega_{+}. \]
Theorem (Green’s Representation Theorem)

\[ u(x) = u_{\text{inc}}(x) - \int_{\Gamma} \Phi(x, y) \partial_{n}^{+} u(y) \, ds(y), \quad x \in \Omega_{+}. \]

Taking a linear combination of Dirichlet \((\gamma_{+})\) and Neumann \((\partial_{n}^{+})\) traces, we get the **boundary integral equation** (Burton & Miller 1971)

\[
\frac{1}{2} \partial_{n}^{+} u(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} - i \eta \Phi(x, y) \right) \partial_{n}^{+} u(y) \, ds(y) = F(x), \quad x \in \Gamma,
\]

where

\[ F := \partial_{n}^{+} u_{\text{inc}} - i \eta \gamma_{+} u_{\text{inc}}. \]
\[ \Delta u + k^2 u = 0 \]

\[ \Gamma u = 0 \]

\[ u - u^{inc} \text{ satisfies SRC} \]

\[ \Omega_+ \]

\[ \Omega_- \]

\[ \frac{1}{2} \partial^+_n u(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} - i\eta \Phi(x, y) \right) \partial^+_n u(y) ds(y) = F(x), \quad x \in \Gamma, \]

in operator form

\[ A_{k,\eta} \partial^+_n u = F := \partial^+_n u^{inc} - i\eta \gamma_+ u^{inc}. \]
\[ \Delta u + k^2 u = 0 \]

\[ \Gamma \ u = 0 \]

\[ u - u^{\text{inc}} \] satisfies SRC

\[ \frac{1}{2} \partial_n^+ u(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} - i \eta \Phi(x, y) \right) \partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma, \]

in operator form

\[ A_{k, \eta} \partial_n^+ u = F := \partial_n^+ u^{\text{inc}} - i \gamma_+ u^{\text{inc}}. \]

**Theorem** *(Burton & Miller 1971, Mitrea 1996, C-W & Langdon 2007)*

*If \( \eta \in \mathbb{R}, \eta \neq 0, \) then this integral equation is uniquely solvable in \( L^2(\Gamma). \)*
\[ \Delta u + k^2 u = 0 \]

\[ \Gamma u = 0 \]

\[ u - u^{inc} \text{ satisfies SRC} \]

\[ \frac{1}{2} \partial^+_n u(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} - i \eta \Phi(x, y) \right) \partial^+_n u(y) ds(y) = F(x), \quad x \in \Gamma, \]

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The standard choice is \( \eta = k \), and with this choice we have

\[ \| A_{k, k}^{-1} \|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \lesssim 1 \]

if \( \Omega_- \) is **star-shaped** (C-W, Monk 2008) or \( C^\infty \) and **nontrapping** (Baskin, Spence, Wunsch 2016).
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if \( \Omega_- \) is star-shaped (C-W, Monk 2008) or \( C^\infty \) and nontrapping (Baskin, Spence, Wunsch 2016). Where does this bound come from and what if \( \Omega_- \) is trapping?
\[ \Delta u + k^2 u = f \in L^2(\Omega_+), \text{ compactly supported} \]

\[ \Gamma u = g \in H^1(\Gamma) \]

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**A recipe for bounding** \( \| A_{k,k}^{-1} \| \) (Baskin, Spence, Wunsch 2016, C-W, Spence, Gibbs, Smyshlyaev 2020)
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**Step 1** (Resolvent Estimate). Show that, for every \( R > 0 \), if \( g = 0 \), \n
\[ \| u \|_{L^2(\Omega_R)} \lesssim c(k) \| f \|_{L^2(\Omega_+)} , \]

where \( \Omega_R := \{ x \in \Omega_+ : |x| < R \} \).
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\[ A_{k,k}^{-1} = I - (DtN_{k,k}^+ - i k) ItD_{k}^- \]

and bounding \( ItD_{k}^- \) as in Baskin, Spence, Wunsch (2016)
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\[ \| A_{k,k}^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim k c(k) \]

if each component of \( \Omega_- \) is star-shaped or \( C^\infty \).
\[ \Delta u + k^2 u = f \in L^2(\Omega_+), \text{ compactly supported} \]

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\[ \| A_{k,k}^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim k^{3/2} c(k) \]

for general Lipschitz \( \Omega_- \).
Recap of resolvent estimates

\[ \|u\|_{L^2(\Omega_R)} \lesssim c(k) \|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0, \]

where \( c(k) = k^{-1} \) for nontrapping obstacles, and

\[
\begin{align*}
c(k) &= \exp(\alpha k) \quad &\text{elliptic} & \& \ C^\infty \\
c(k) &= k^{-1} \log(2 + k) \quad &\text{hyperbolic} & \& \ C^\infty \\
c(k) &= k \quad &\text{parabolic}
\end{align*}
\]

Further, for all Lipschitz \( O \) and all \( \delta, \varepsilon > 0 \), there exists \( E \) with \( |E| \leq \varepsilon \), such that

\[ c(k) = k^{5d/2+\delta}, \quad k \in [k_0, \infty) \setminus E. \]
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Applying our general recipe

\[ \|A_{k,k}^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \lesssim k^{3/2} c(k), \]

in all these cases, indeed \( \|A_{k,k}^{-1}\| \lesssim kc(k) \) if \( O \) is \( C^\infty \).
1st Kind BIE (see Siavash Sadeghi’s poster for details)

\[ \Delta u + k^2 u = 0 \]

\[ \Omega_+ \]

\[ \Omega_- \]

\[ u - u^{\text{inc}} \text{ satisfies SRC} \]

\[ u(x) = u^{\text{inc}}(x) - \int_{\Gamma} \Phi(x, y) \partial_n^+ u(y) \, ds(y), \quad x \in \Omega_+. \]

Theorem (Green’s Representation Theorem)
1st Kind BIE (see Siavash Sadeghi’s poster for details)

\[ \Delta u + k^2 u = 0 \]

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**Theorem (Green’s Representation Theorem)**

\[
 u(x) = u^{\text{inc}}(x) - \int_{\Gamma} \Phi(x, y) \partial_n^+ u(y) \, ds(y), \quad x \in \Omega_+. 
\]

Taking the Dirichlet (\(\gamma_+\)) trace we get the 1st kind boundary integral equation

\[
 \int_{\Gamma} \Phi(x, y) \partial_n^+ u(y) \, ds(y) = F(x) := \gamma_+ u^{\text{inc}}(x), \quad x \in \Gamma, 
\]
\[\Delta u + k^2 u = 0\]

\[\Gamma u = 0\]

\[u - u^{inc}\] satisfies SRC

\[
\int_{\Gamma} \Phi(x, y) \partial^+_n u(y) ds(y) = F(x), \quad x \in \Gamma,
\]

in operator form

\[S_k \partial^+_n u = F := \gamma_+ u^{inc}.\]
\[ \mathcal{N} u^{inc} \quad \Delta u + k^2 u = 0 \]

\[ \Gamma u = 0 \]

\[ u - u^{inc} \text{ satisfies SRC} \]

\[ \int_{\Gamma} \Phi(x, y) \partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma, \]

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\[ S_k \partial_n^+ u = F := \gamma_+ u^{inc}. \]

It is standard that \( S_k \) invertible iff \( k^2 \not\in \text{spec}(-\Delta_D(\Omega_-)) \).
\[ \Delta u + k^2 u = 0 \]

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\[ S_k^{-1} = DtN_k^- - DtN_k^+. \]
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Further, by our previous recipe, \( \|DtN_k^+\| \lesssim kc(k) \).
\[ \nabla u^{\text{inc}} \quad \Delta u + k^2 u = 0 \]

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\[ \Omega_+ \quad \Omega_- \]

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\[ \|DtN_k^-\| \lesssim kc_-(k) \]
\( \mathcal{H} \ u^{\text{inc}} \quad \Delta u + k^2 u = 0 \)

\( \Gamma \ u = 0 \)

\( \Omega_+ \)

\( \Omega_- \)

\( u - u^{\text{inc}} \) satisfies SRC

\[ \int_{\Gamma} \Phi(x, y) \partial_n^+ u(y) ds(y) = F(x), \quad x \in \Gamma, \]

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\[ \|DtN_k^-\| \lesssim k c_-(k) \lesssim \frac{k}{\text{dist}(k^2, \text{spec}(-\Delta_D(\Omega_-)))}, \]

using that \(-\Delta_D(\Omega_-)\) is self-adjoint, so that

\[ c_-(k) = \|(\Delta_D + k^2)^{-1}\|_{L^2(\Omega_-) \rightarrow L^2(\Omega_-)} = \left[ \text{dist}(k^2, \text{spec}(-\Delta_D(\Omega_-))) \right]^{-1}. \]
Comparing 1st & 2nd Kind BIEs: general Lipschitz $\Omega_-$

\[ \nabla \cdot u^{\text{inc}} \quad \Delta u + k^2 u = 0 \]
\[ \Gamma u = 0 \]
\[ u - u^{\text{inc}} \text{ satisfies SRC} \]

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$\Omega_-$ \hspace{1cm} $\Omega_+$

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The bounds in the last row hold for $k \in [k_0, \infty) \setminus E$, with $|E|$ and $\delta > 0$ arbitrarily small; for the $S_k^{-1}$ bound see Siavash’s poster.
Comparing 1st & 2nd Kind BIEs: general Lipschitz $\Omega_-$

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Conjecture.

$$\|DtN_k^+\|, \|A_{k,k}^{-1}\|, \|S_k^{-1}\| \lesssim k^d \log^2(2 + k), \quad k \in [k_0, \infty) \setminus E.$$
Overview of Talk

1 What is this talk about?

2 A key tool: resolvent estimates
   - What are they?
   - The known estimates and their geometries

3 Applications to Boundary Integral Equations
   - The standard Burton-Miller 2nd kind BIE
   - The standard 1st kind BIE
   - A new 1st kind IE

4 Conclusions
1st kind IE for general compact $O$ (Caetano et al 2024)

\[ \nabla u^{inc} \quad \Delta u + k^2 u = 0 \]
\[ u = 0 \]
\[ u - u^{inc} \text{ satisfies SRC} \]

Recall $O$ is compact and $\Omega := \mathbb{R}^d \setminus O$ is connected, and assume $u^{inc} \in H^{1,\text{loc}}(\mathbb{R}^d)$. 
1st kind IE for general compact $O$ (Caetano et al 2024)

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$$\partial \Omega = \partial O \subset \Gamma \subset O.$$
1st kind IE for general compact $O$ (Caetano et al 2024)

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**Step 2.** Look for a solution in the form $u = u^{\text{inc}} + A\phi$, for some $\phi \in H^{-1}(\mathbb{R}^d)$ supported on $\Gamma$, where

$$A\psi(x) := \int_{\mathbb{R}^d} \Phi(x, y)\psi(y) \, dy,$$

for $\psi \in L^2_{\text{comp}}(\mathbb{R}^d), \ x \in \mathbb{R}^d$. 

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**Step 1.** Choose a compact \( \Gamma \) such that

\[ \partial \Omega = \partial O \subset \Gamma \subset O. \]

**Step 2.** Look for a solution in the form \( \mathbf{u} = \mathbf{u}^{\text{inc}} + A\phi \), for some \( \phi \in H^{-1}(\mathbb{R}^d) \) supported on \( \Gamma \), where

\[ A\psi(x) := \int_{\mathbb{R}^d} \Phi(x,y)\psi(y) \, dy, \quad \text{for} \quad \psi \in L^2_{\text{comp}}(\mathbb{R}^d), \ x \in \mathbb{R}^d. \]

**Step 3.** Enforce \( \mathbf{u} = 0 \) on \( \partial \Omega \) by requiring \( \mathbf{u} = 0 \) on \( \Gamma \), in the sense that, where \( \Omega^* := \mathbb{R}^d \setminus \Gamma \) and \( \chi \in C_0^\infty(\mathbb{R}^d) \) with \( \chi = 1 \) near \( \Gamma \),

\[ \chi \mathbf{u} \in \widetilde{H}^1(\Omega^*) := \overline{C_0^\infty(\Omega^*)}^{H^1(\mathbb{R}^d)}. \]

In other words, we require, where \( P : H^1(\mathbb{R}^d) \to \widetilde{H}^1(\Omega^*)^\perp \) is orthogonal projection, that

\[ P(\chi \mathbf{u}) = 0 \]
\[ \nabla u^{\text{inc}} \quad \Delta u + k^2 u = 0 \]

\[ \Gamma u = 0 \quad \Omega \quad u - u^{\text{inc}} \text{ satisfies SRC} \]

Recall $O$ is compact and $\Omega := \mathbb{R}^d \setminus O$ is connected, and assume $u^{\text{inc}} \in H^{1,\text{loc}}(\mathbb{R}^d)$.

**Step 1.** Choose a compact $\Gamma$ such that

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\[ P(\chi u) = 0 \quad \Leftrightarrow \quad S_k \phi = g := -P(\chi u^{\text{inc}}), \quad \text{where} \quad S_k \phi := P(\chi A\phi). \]
\( S_k \phi = g \) can be solved by Galerkin BEM (Caetano et al. 2024)
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**Example computation:** $\Gamma = O = \partial O = \text{Sierpinski tetrahedron,}$

$$u = u^{\text{inc}} + A\phi,$$

where $\phi \in H^{-1}_\Gamma := \{ \psi \in H^{-1}(\mathbb{R}^d) : \text{supp} (\psi) \subset \Gamma \}$ satisfies

$$S_k \phi = g := -P(\chi u^{\text{inc}}),$$

where $S_k \phi := P(\chi A\phi)$.

Plotted is the **scattered field** $A\phi$. 
Recap. $\Gamma$ is compact with $\partial \Omega = \partial O \subset \Gamma \subset O$. Look for solution as $u = u^{\text{inc}} + A\phi$, where

$$A\psi(x) := \int_{\mathbb{R}^d} \Phi(x,y)\psi(y) \, dy,$$

for $\psi \in L^2_{\text{comp}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $\phi \in H^{-1}(\mathbb{R}^d)$ is supported on $\Gamma$, i.e., $\phi \in H^{-1}_\Gamma \subset H^{-1}(\mathbb{R}^d)$. 

$$\iota \quad u^{\text{inc}} \quad \Delta u + k^2 u = 0 \quad \Gamma u = 0 \quad \Omega$$

$u - u^{\text{inc}}$ satisfies SRC.
\[ \Delta u + k^2 u = 0 \]

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**Recap.** \( \Gamma \) is compact with \( \partial \Omega = \partial O \subset \Gamma \subset O \). Look for solution as \( u = u^{\text{inc}} + A\phi \), where

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and \( \phi \in H^{-1}(\mathbb{R}^d) \) is supported on \( \Gamma \), i.e., \( \phi \in H^{-1}_\Gamma \subset H^{-1}(\mathbb{R}^d) \). This satisfies the scattering problem iff

\[ \mathbf{S}_k \phi = g := -P(\chi u^{\text{inc}}), \quad \text{where } \mathbf{S}_k \phi := P(\chi A\phi), \]

\( \Omega^* := \mathbb{R}^d \setminus \Gamma \), and \( P : H^1(\mathbb{R}^d) \to \widetilde{H}^1(\Omega^*)^\perp = (H^{-1}_\Gamma)^* \) is orthogonal projection.
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Theorem (Caetano et al 2024, C-W & Sadeghi 2024+)

Let \( \Omega_- := O \setminus \Gamma \), and \( c(k) \) and \( c_-(k) \) denote the bounds in the resolvent estimates for \( \Omega \) and \( \Omega_- \). Then \( S_k : H^{-1}_{\Gamma} \to (H^{-1}_{\Gamma})^* \) is invertible iff \( k^2 \not\in \text{spec}(\Delta_D(\Omega_-)) \), and

\[
\|S_k^{-1}\| \lesssim k^2 c(k) + k^2 c_-(k)
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**Theorem (Caetano et al 2024, C-W & Sadeghi 2024+)**

Let $\Omega_- := O \setminus \Gamma$, and $c_+(k)$ and $c_-(k)$ denote the bounds in the resolvent estimates for $\Omega$ and $\Omega_-$. Then $S_k : H^{-1}_\Gamma \to (H^{-1}_\Gamma)^*$ is invertible iff $k^2 \not\in \text{spec}(-\Delta_D(\Omega_-))$, and

$$\|S_k^{-1}\| \lesssim k^2 c_+(k) + k^2 c_-(k) \lesssim k^{(5d+4)/2+\delta}$$

for $k \in [k_0, \infty) \setminus E$ with $|E| \leq \varepsilon$, if $\Omega$ is Lipschitz.
Recap. $\Gamma$ is compact with $\partial \Omega = \partial O \subset \Gamma \subset O$. Look for solution as $u = u^{inc} + A\phi$, where

$$A\psi(x) := \int_{\mathbb{R}^d} \Phi(x,y)\psi(y) \, dy, \quad \text{for} \quad \psi \in L^2_{\text{comp}}(\mathbb{R}^d), \ x \in \mathbb{R}^d,$$

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$$S_k\phi = g := -P(\chi u^{inc}), \quad \text{where} \quad S_k\phi := P(\chi A\phi),$$

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Conjecture

Let $\Omega_- := O \setminus \Gamma$, and $c(k)$ and $c_-(k)$ denote the bounds in the resolvent estimates for $\Omega$ and $\Omega_-$. Then

$$\|S_k^{-1}\| \lesssim kc(k) + kc_-(k) \lesssim k^{d+1} \log^2(2 + k),$$

for $k \in [k_0, \infty) \setminus E$ with $|E| \leq \varepsilon$, for every obstacle $O$. 


Conclusions

In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles, including:
  - elliptic
  - hyperbolic
  - parabolic

How resolvent estimates lead in a "black box" way to:

- Bounds on (exterior and interior) DtN maps
- \( \|A^{-1}_{k,k}\|, \|S^{-1}_{k}\|, \|S^{-1}_{k}\| \)

Lots of open problems:

- Our conjectures above;
- Are all our estimates for integral operators in terms of resolvent estimates sharp?
- Resolvent estimates are missing, or need sharpening, for many configurations, notably where the obstacle is non-smooth, e.g. Lipschitz or fractal.
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In this talk you have seen:

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Example open problems

**Thin curved screen:** resolvent estimate? Bound on $\|S_k^{-1}\|$?

**Fractal:** resolvent estimate? Sharp bound on $\|S_k^{-1}\|$?
Conclusions

In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles, including elliptic, hyperbolic, parabolic
- The Morawetz/Rellich identity method for proving these estimates
- The standard 1st and 2nd kind BIEs when $O$ is Lipschitz, and a new 1st kind integral equation for general compact $O$
- How resolvent estimates lead in a “black box” way to:
  - bounds on (exterior and interior) DtN maps
  - bounds on $\|A_{k,k}^{-1}\|, \|S_{k}^{-1}\|, \|S_{k}^{-1}\|

Lots of open problems:

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References


