Semiclapp Summer School

Solutions to exercises: WKB solutions and Propagation of singularities

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1 WKB solutions

Solution to Exercise 1

1. We plug the ansatz into the equation $-\frac{h^2}{2}\Delta u + Vu = 0$ and group terms according to powers of h. The $O(h^0)$ terms give the Eikonal equation. The $O(h)$ terms give the transport equation. See the lecture notes or [1, Chapter 2].

2. For any $x \in \{x_1 = 0\}$, the o.d.e.
$$\frac{d}{dt}x(t) = \nabla\phi\left(x(t)\right)$$
with initial datum $x(t) = x$ can be solved locally by the Cauchy–Lipschitz theorem, locally uniformly with respect to x. Then it holds for any $a \in C^\infty(\mathbb{R}^d)$ and $x \in \{x_1 = 0\}$:
$$\frac{d}{dt}a\left(S(t)x\right) = \left(\nabla\phi \cdot \nabla a\right)\left(S(t)x\right) = -\frac{1}{2}(a\Delta\phi)\left(S(t)x\right),$$
from which we deduce that
$$a\left(S(t)x\right) = a(x)\exp\left(-\frac{1}{2}\int_0^t (\Delta\phi)\left(S(t)x\right)\right).$$
Therefore, solutions a_0 to the transport equation with initial datum $a_{0|x_1=0} = b \in C^\infty(\mathbb{R}^{d-1})$ can be written
$$a_0(S(t)x) = b(x)\exp\left(-\frac{1}{2}\int_0^t (\Delta\phi)\left(S(t)x\right)\right), \qquad x \in \{x_1 = 0\}.$$

Solution to Exercise 2

1. We change variables to check that $\|u_h\|_{L^2} = \|u\|_{L^2}$. Setting $h = E^{-1}$, we have
$$-\frac{h^2}{2}u_h''(x) + \frac{x^2}{2}u_h(x) = -\frac{h}{2}u''(h^{-1/2}x) + h\frac{(h^{-1/2}x)^2}{2}u(h^{-1/2}x) = hEu(h^{-1/2}x) = u_h.$$

2. The trajectory of the Hamiltonian flow starting from (x_0, ξ_0) is the map $t \mapsto (x^t, \xi^t)$ satisfying
$$\frac{d}{dt}\begin{pmatrix} x^t \\ \xi^t \end{pmatrix} = \mathsf{H}\begin{pmatrix} x^t \\ \xi^t \end{pmatrix} = \begin{pmatrix} \xi^t \\ -x^t \end{pmatrix}, \qquad \begin{pmatrix} x^0 \\ \xi^0 \end{pmatrix} = \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix}$$
In particular, it holds $\ddot{x}^t = -x^t$. This is solved by linear combinations of $\cos t$ and $\sin t$. Matching the initial data, we obtain
$$\begin{pmatrix} x^t \\ \xi^t \end{pmatrix} = \begin{pmatrix} x_0\cos t + \xi_0 \sin t \\ -x_0 \sin t + \xi_0 \cos t \end{pmatrix}.$$

3. Let ν_E be the uniform probability measure on the circle of radius $\sqrt{2E}$. Then the classical probability density at energy E is the pushforward measure $\pi_*\nu_E$, where $\pi(x,\xi) = x$ is the cotangent bundle projection. Then for any $f \in C_c^0(\mathbb{R})$, we have
$$\int_{\mathbb{R}} f(x)\, d\pi_*\nu_E(x) = \int_{\sqrt{2E}\mathbb{S}^1} (f\circ\pi)(\theta)\, d\nu_E(\theta) = (2\pi)^{-1}\int_0^{2\pi} f\left(\sqrt{2E}\cos t\right)\, dt.$$

We use the change of variables $y = \sqrt{2E}\cos t$ whose Jacobian is $\sqrt{2E}|\sin t| = \sqrt{2E - y^2}$ to obtain

$$\int_{\mathbb{R}} f(x)\, d\pi_* \nu_E(x) = \pi^{-1} \int_{-\sqrt{2E}}^{\sqrt{2E}} f(y) \frac{dy}{\sqrt{2E - y^2}}.$$

4. Taking square roots, the Eikonal equation reads

$$\phi'(x) = \pm\sqrt{2E - x^2},$$

which in turn gives

$$\phi(x) = C \pm \int_0^x \sqrt{2E - y^2}\, dy.$$

It remains to compute the integral. We set the change of variables $\sin t = y/\sqrt{2E}$, for which we have $\sqrt{2E - y^2}\, dt = dy$, and therefore

$$\begin{aligned}\int_0^x \sqrt{2E - y^2}\, dy &= \int_0^{\arcsin \frac{x}{\sqrt{2E}}} 2E\cos^2 t\, dt = E \arcsin \frac{x}{\sqrt{2E}} + E \int_0^{\arcsin \frac{x}{\sqrt{2E}}} \cos(2t)\, dt \\ &= E \arcsin \frac{x}{\sqrt{2E}} + \frac{E}{2} \sin\left(2 \arcsin \frac{x}{\sqrt{2E}}\right) = E \arcsin \frac{x}{\sqrt{2E}} + E\frac{x}{\sqrt{2E}} \cos\left(\arcsin \frac{x}{\sqrt{2E}}\right) \\ &= E \arcsin \frac{x}{\sqrt{2E}} + E\frac{x}{\sqrt{2E}} \sqrt{1 - \frac{x^2}{2E}},\end{aligned}$$

which concludes the proof.

5. Note that $\phi'(x) = \sqrt{2E - x^2}$ and $\phi''(x) = -x(2E - x^2)^{-1/2}$ so that the transport equation reduces to

$$a_0'(x) = \frac{x/2}{2E - x^2} a_0(x).$$

The map $\frac{1}{4}\log|2E - x^2|$ is a primitive of the factor in the right-hand side, so that solutions to this o.d.e. are of the desired form. We observe that the modulus squared of the amplitude a_0 gives back the classical probability density. This also matches the profile of the amplitude on Figure 1.

6. We change variables by setting $X = \pm(x - x_\pm)/\alpha$ for some $\alpha > 0$ to be determined. Writing $U_h(X) = u_h(\pm\alpha X + x_\pm) = u_h(x)$, we have

$$U_h''(X) = \alpha^2 u_h''(\pm\alpha X + x_\pm) = \alpha^2 \frac{2}{h^2} \sqrt{2E} \alpha X u_h(\pm\alpha X + x_\pm) = \alpha^3 \frac{2^{3/2} E^{1/2}}{h^2} X U_h(X).$$

Choosing $\alpha = 2^{-1/2} E^{-1/6} h^{2/3}$, we see that U_h solves the Airy equation. Rewriting $u_h(x) = U_h(X) = \mathrm{Ai}(X)$ in terms of the x variable yields the desired result. This holds for $x = x_\pm + O(h^{3/2})$ (the quadratic error term in the Taylor expansion of the potential is then much smaller than the values of $U_h(X)$).

7. (If time allows:) The above computations generalize to smooth confining potentials with a unique critical point for instance, although the solutions to the Eikonal and transport equations are not as explicit as for the harmonic oscillator (they involve a primitive of $\sqrt{2E - V(x)}$...).

Solution to Exercise 3

We refer to [2, Proposition 3.25] or [3, Theorem 4.15].

2 Semiclassical wave front set and defect measures

Solution to Exercise 4

- Let us show that $\mathrm{WF}_h e^{\frac{i}{h}\alpha\cdot x} = \mathbb{R}^d \times \{\alpha\}$. Let ϕ be a cut-off function. Then it holds

$$\mathcal{F}_h\left(\phi(x)e^{\frac{i}{h}\alpha\cdot x}\right)(\xi) = \mathcal{F}_h\phi(\xi-\alpha) = h^{-d/2}\mathcal{F}_1\phi\left(\frac{\xi-\alpha}{h}\right). \tag{2.1}$$

  Now since ϕ is smooth, its Fourier transform has rapid decay, namely

$$\forall n\in\mathbb{N}, \exists C_n>0: \qquad \left|\mathcal{F}_h\left(\phi(x)e^{\frac{i}{h}\alpha\cdot x}\right)(\xi)\right| \le C_n h^{-d/2}\left(1+\frac{|\xi-\alpha|}{h}\right)^{-n} = C_n\frac{h^{n-d/2}}{(h+|\xi-\alpha|)^n}.$$

  For ξ in a sufficiently small neighborhood of $\xi_0 \neq \alpha$, the denominator is bounded from below by a positive constant, so we get $O(h^\infty)$, namely $(x_0,\xi_0)\notin \mathrm{WF}_h e^{\frac{i}{h}\alpha\cdot x}$ for any $x_0\in\mathbb{R}^d$. This proves that $\mathrm{WF}_h e^{\frac{i}{h}\alpha\cdot x}\subset \mathbb{R}^d\times\{\alpha\}$. To prove the converse inclusion, we observe that taking $\xi=\xi_0=\alpha$ in (2.1), we have

$$\mathcal{F}_h\left(\phi(x)e^{\frac{i}{h}\alpha\cdot x}\right)(\xi) = h^{-d/2}\mathcal{F}_1\phi(0) = (2\pi h)^{-d/2}\int_{\mathbb{R}^d}\phi(x)\,dx,$$

  which is not $O(h^\infty)$ for general ϕ.

- The proof is quite similar for $\mathrm{WF}_h a$: for any cut-off function ϕ, we have

$$|\mathcal{F}_h(\phi a)(\xi)| = h^{-d/2}\left|\mathcal{F}_1(\phi a)\left(\frac{\xi}{h}\right)\right| \le C h^{-d/2}\frac{h^n}{(h+|\xi|)^n}. \tag{2.2}$$

  In a neighborhood of any $\xi_0\neq 0$, this is $O(h^\infty)$ using the rapid decay of $\mathcal{F}_1(\phi a)$. In addition, if ϕ is supported away from the support of a, this is zero for all ξ. This shows that $\mathrm{WF}_h a\subset \mathrm{supp}\ a\times\{0\}$. Now if ϕ is supported in the open set $\{a>0\}$, then from (2.2), $|\mathcal{F}_h(\phi a)(\xi)|$ is not $O(h^\infty)$ for general ϕ, which shows that $\{a>0\}\times\{0\}\subset \mathrm{WF}_h a$. Yet the wave front set is closed and $\overline{\{a>0\}} = \mathrm{supp}\ a$, so we finally conclude that $\mathrm{WF}_h a = \mathrm{supp}\ a\times\{0\}$.

Solution to Exercise 5

Let $P_j := \frac{h}{i}\partial_j - \partial_j\phi(x) = \mathrm{Op}_h(p_j)$ with $p_j(x,\xi)=\xi_j-\partial_j\phi(x)$. Let $(x_0,\xi_0)\notin \mathrm{Graph}(d\phi) = \{\xi = d\phi(x)\}$. Then there exists an index j such that $p_j(x_0,\xi_0)\neq 0$. Since $P_j e^{\frac{i}{h}\phi} = 0 = O(h^\infty)$, we obtain by definition of the wave front set that $\mathrm{WF}_h e^{\frac{i}{h}\phi}\subset \mathrm{Graph}(d\phi)$. The converse inclusion follows from Exercise 7 (in fact stationary phase).

We refer to [3, Examples pp. 189-190] for more examples.

Solution to Exercise 6

Denote by $\pi: T^\star\mathbb{R}^d\to\mathbb{R}^d$ the cotangent bundle projection $\pi(x,\xi)=x$.

1. (bump function going to infinity) The sequence $(u_h)_h$ classically converges weakly in L^2 to 0. In particular, the pushforward measure $\pi_*\mu$ vanishes, so $\mu=0$ is the unique semiclassical measure associated with $(u_h)_h$.

2. (Gaussian wave packet) We let $a\in\mathcal{S}(\mathbb{R}^d)$ and compute the inner product

$$\begin{aligned}(u_h,\mathrm{Op}_h(a)u_h)_{L^2} &= (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}}\bar{u}_h(x)e^{\frac{i}{h}\xi.(x-y)}a(x,\xi)u_h(y)\,dyd\xi dx\\ &= (2\pi h)^{-d}(\pi h)^{-d/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}}\exp\left(-\frac{|x-x_0|^2+|y-x_0|^2}{2h}\right)e^{\frac{i}{h}(\xi-\xi_0).(x-y)}a(x,\xi)\,dyd\xi dx.\end{aligned}$$

We compute the integral over y:

$$(2\pi)^{-d/2}\int_{\mathbb{R}^d}(\pi h)^{-d/4}\exp\left(-\frac{|y-x_0|^2}{2h}\right)e^{-\frac{i}{h}(\xi-\xi_0).y}\,dy = e^{-\frac{i}{h}(\xi-\xi_0).x_0}\mathcal{F}_1\left((\pi h)^{-d/4}e^{-\frac{|\bullet|^2}{2h}}\right)\left(\frac{\xi-\xi_0}{h}\right)$$
$$= (\pi/h)^{-d/4}e^{-\frac{i}{h}(\xi-\xi_0).x_0}\exp\left(-\frac{|\xi-\xi_0|^2}{2h}\right).$$

We finally obtain

$$(u_h,\mathrm{Op}_h(a)u_h)_{L^2} = (2\pi)^{-d/2}h^{-d}\pi^{-d/2}\int_{\mathbb{R}^{2d}}\exp\left(-\frac{|x-x_0|^2+|\xi-\xi_0|^2}{2h}\right)e^{\frac{i}{h}(\xi-\xi_0).(x-x_0)}a(x,\xi)\,dxd\xi.$$

We can rewrite this as

$$(u_h,\mathrm{Op}_h(a)u_h)_{L^2} = 2^{d/2}\int_{\mathbb{R}^{2d}}e^{\frac{i}{h}(\xi-\xi_0).(x-x_0)}a(x,\xi)\,d\mu_h(x,\xi)$$

where

$$d\mu_h(x,\xi) = (2\pi h)^{-d}\exp\left(-\frac{|x-x_0|^2+|\xi-\xi_0|^2}{2h}\right)\,dxd\xi$$

is a probability measure. The sequence of measures μ_h classically converges to $\delta_{(x_0,\xi_0)}$ weakly. Therefore

$$(u_h,\mathrm{Op}_h(a)u_h)_{L^2} = Ca(x_0,\xi_0)+o(1)$$

as $h\to 0$, where

$$C := 2^{d/2}\int_{\mathbb{R}^{2d}}e^{\frac{i}{h}(\xi-\xi_0).(x-x_0)}d\mu_h(x,\xi) = 2^{d/2}\int_{\mathbb{R}^{2d}}e^{i\xi.x}(2\pi)^{-d}\exp\left(-\frac{|x|^2+|\xi|^2}{2}\right)\,dxd\xi$$

is independent of h by changing variables. Yet for $a\equiv 1$, we have

$$1 = \|u_h\|_{L^2}^2 = C+o(1),$$

so that finally

$$(u_h,\mathrm{Op}_h(a)u_h)_{L^2}\xrightarrow{h\to 0}a(x_0,\xi_0),$$

that is to say $\mu=\delta_{(x_0,\xi_0)}$.

3. (plane wave) We compute for any $a\in\mathcal{S}(\mathbb{R}^{2d})$:

$$\langle\bar{u}_h,\mathrm{Op}_h(a)u_h\rangle_{\mathcal{S}',\mathcal{S}(\mathbb{R}^{2d})} = (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}}e^{\frac{i}{h}((\xi-h^{1-\beta}\alpha).(x-y)}a(x,\xi)\,dyd\xi dx = a(x,h^{1-\beta}\alpha),$$

where we used the Fourier inversion formula. Therefore

$$\langle\bar{u}_h,\mathrm{Op}_h(a)u_h\rangle_{\mathcal{S}',\mathcal{S}(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}}a(x,\xi)\,d\delta_{\mathbb{R}^d\times\{h^{1-\beta}\alpha\}},$$

where $\delta_{\mathbb{R}^d\times\{h^{1-\beta}\alpha\}}$ is the Lebesgue measure on $\mathbb{R}^d\times\{h^{1-\beta}\alpha\}$. Examining the weak limit of these measures as $h\to 0$, we obtain $\mu=\delta_{\mathbb{R}^d\times\{0\}}$ if $\beta<1$, $\mu=\delta_{\mathbb{R}^d\times\{\alpha\}}$ if $\beta=1$ and $\mu=0$ if $\beta>1$.

We refer to [3, Examples pp. 102-104] for more examples.

Solution to Exercise 7

We follow [3, Example 2 p. 103].

1. We have $P_je^{\frac{i}{h}\phi}=0=o(1)$ as $h\to 0$, with $P_j=\frac{h}{i}\partial_j-\partial_j\phi(x)=\mathrm{Op}_h(\xi_j-\partial_j\phi(x))$. Therefore, Proposition 2 gives supp $\mu\subset\bigcap_j\{\xi_j=\partial_j(x)\}=\mathrm{Graph}(d\phi)$. This can also be seen as a consequence of the fact that $\mathrm{WF}_he^{\frac{i}{h}\phi}\subset\mathrm{Graph}(d\phi)$.

2. (a) Symbols of an operator A_h for the left, right or Weyl quantizations differ from a $O(h)$ term, which does not contribute to the semiclassical defect measure.

(b) We compute

$$(u_h, a(x,hD)u_h)_{L^2} = (2\pi h)^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} a(x,\xi) e^{\frac{i}{h}(\xi.(x-y) - \phi(x) + \phi(y)} \overline{b(x)} b(y)\, dy d\xi \right) dx.$$

(c) The critical points satisfy $\partial_\xi \Phi_x = x - y = 0$ and $\partial_y \Phi_x = d\phi(y) - \xi$, namely $(y,\xi) = (x, d\phi(x))$. The Hessian matrix reads

$$\partial^2 \Phi_x(y,\xi) = \begin{pmatrix} \partial^2\phi(y) & -1 \\ -1 & 0 \end{pmatrix}.$$

Its determinant is always ± 1 (because $\partial^2\Phi_x \partial^2 \Phi_x^\top$ is upper triangular with 1 on the diagonal for instance). To compute the signature, we can argue that it is constant along continuous non-singular deformations of the matrix, since it is integer-valued. For instance, take

$$t \longmapsto \begin{pmatrix} t\partial^2\phi(y) & -1 \\ -1 & 0 \end{pmatrix},$$

which is non-singular for every $t \in [0,1]$, coincides with $\partial^2\Phi_x(y,\xi)$ at $t=1$ and has signature 0 for $t=0$ (the spectrum is made of two eigenvalues, ± 1, with multiplicity d).

(d) Proposition 3 gives

$$\int_{\mathbb{R}^{2d}} a(x,\xi) e^{\frac{i}{h}\Phi_x(y,\xi)} \overline{b(x)} b(y)\, dy d\xi \xrightarrow{h\to 0} a(x, d\phi(x)) |b(x)|^2,$$

which gives exactly the sought result.

Exercise 8: Back to the 1D harmonic oscillator.

Let $(u_h)_{h\in(0,1]}$ satisfies $P_h u_h = o_{L^2}(h)$ with $P_h = -\frac{h^2}{2}\Delta + \frac{|x|^2}{2} - 1$, any semiclassical defect measure μ is supported in $\{\frac{x^2+\xi^2}{2} = 1\}$ and is invariant by the Hamiltonian flow. Therefore μ is a multiple of the uniform measure on the circle of radius $\sqrt{2}$. It is a probability measure (done in the lecture notes–use factorization from Melissa Tacy's lecture).

References

[1] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[2] Stéphane Nonnenmacher. Introduction to semiclassical analysis. Lecture notes, `https://www.imo.universite-paris-saclay.fr/~stephane.nonnenmacher/enseign/Cours-Semiclassical-Analysis2019-lecture1-8.pdf`.

[3] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.