Semiclapp Summer School

Exercise session: WKB solutions and Propagation of singularities

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1 WKB solutions

Exercise 1: WKB ansatz in dimension $d \geq 1$

Let $V \in C^\infty(\mathbb{R}^d)$ and consider the equation

$$-\tfrac{h^2}{2}\Delta u + V(x)u = 0, \qquad h \in (0,1].$$

1. As was done in Lecture 1, derive from the ansatz $u = (a_0 + ha_1 + \cdots)e^{\frac{i}{h}\phi}$ the first two steps of the WKB approximation, namely the Eikonal equation

$$\frac{1}{2}|\nabla\phi|^2 + V(x) = 0 \tag{1.1}$$

   and the transport equation

$$\nabla\phi \cdot \nabla a_0 + \frac{1}{2}(\Delta\phi)a_0 = 0, \tag{1.2}$$

   with $a_j \in C^\infty(\mathbb{R}^d;\mathbb{C})$ and $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$.

2. Assuming $\partial_{x_1}\phi$ vanishes nowhere on $\{x_1 = 0\}$, solve locally around $\{x_1 = 0\}$ the transport equation (1.2) with initial data on $\{x_1 = 0\}$ in terms of the flow $S(t) : x \mapsto S(t)x$ of the vector field $\nabla\phi$. (Hint: look for an o.d.e. on $t \mapsto a_0(S(t)x)$ for $x \in \{x_1 = 0\}$.)

Exercise 2: 1D harmonic oscillator

Our goal is to understand the asymptotic behavior of eigenfunctions of the semiclassical harmonic oscillator

$$-\frac{h^2}{2}u_h'' + \frac{x^2}{2}u_h = E_h u_h, \qquad E_h \approx 1, \tag{1.3}$$

as $h \to 0$. See Figure 1 for an illustration.

1. Suppose u is such that

$$-\frac{1}{2}u'' + \frac{x^2}{2}u = Eu,$$

   for some $E \gg 1$. Check that $u_h(x) := h^{-1/4}u(h^{-1/2}x)$ satisfies (1.3) with $E_h = 1$ and $\|u_h\|_{L^2} = \|u\|_{L^2}$, for a suitably chosen parameter h.

2. Recall that the Hamiltonian vector field associated with the operator $-\frac{h^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{2}$ is defined by

$$\mathsf{H}(x,\xi) := \xi \cdot \partial_x - x \cdot \partial_\xi.$$

   Draw this vector field in phase space (the (x,ξ) plane) and compute its integral curves (they parametrize the circles $\{\frac{\xi^2}{2} + \frac{x^2}{2} = E\}$).

3. Justify why the classical probability density at energy $E > 0$ (namely the probability to find a classical particle of energy E evolving according to the Hamiltonian flow) is given by

$$x \longmapsto \frac{C}{\sqrt{E - \frac{1}{2}x^2}}\mathbb{1}_{[-\sqrt{2E},+\sqrt{2E}]}(x), \tag{1.4}$$

   for some constant $C > 0$ independent of E. Observe that it is peaked around the turning points of the classical dynamics.

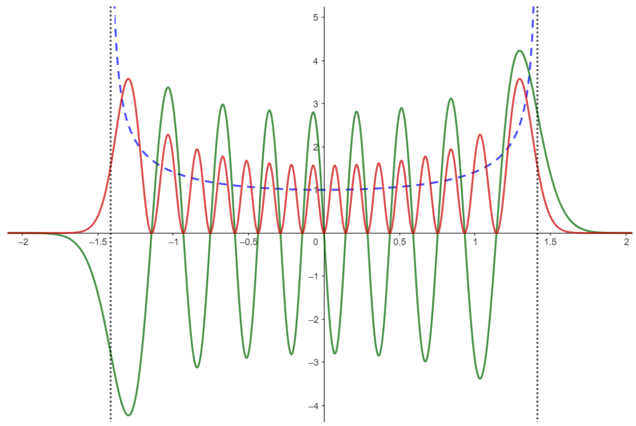

Figure 1: 15th eigenfunction of the harmonic oscillator (green), corresponding probability density (red) and classical probability density (dashed blue line). The classically allowed region lies between the two vertical dotted lines.

4. Show that solutions to the Eikonal equation (1.1) in the classically allowed region $\{\frac{x^2}{2} \leq E\}$, that is to say the support of (1.4), are of the form

$$\phi(x) = C \pm E \left(\arcsin \frac{x}{\sqrt{2E}} + \frac{x}{\sqrt{2E}} \sqrt{1 - \frac{x^2}{2E}} \right).$$

(Hint: one may use a trigonometric change of variables, related to the trajectories of the Hamiltonian flow computed in Question 2.)

5. Still in the classically allowed region, show that solutions to the transport equation (1.2) are of the form

$$a_0(x) = \frac{C}{(2E - x^2)^{1/4}},$$

for an arbitrary constant $C > 0$. Compare with the result of Question 3. Observe that this is consistent with Figure 1.

6. Now we study the behavior of u_h in (1.3) outside the classically allowed region, namely $\{|x| > \sqrt{2E}\}$. Instead of using the WKB expansion, we linearize the potential near the turning points $x_\pm = x_\pm(E) = \pm\sqrt{2E}$:

$$\frac{x^2}{2} = \frac{x_\pm^2}{2} + x_\pm \cdot (x - x_\pm) + \frac{(x - x_\pm)^2}{2} = E \pm \sqrt{2E}(x - x_\pm) + O\left((x - x_\pm)^2\right). \tag{1.5}$$

Pretending that one can neglect the quadratic remainder, (1.3) becomes

$$-\frac{h^2}{2} u_h'' \pm \sqrt{2E}(x - x_\pm) u_h = 0 \tag{1.6}$$

(we write E in place of E_h). We denote by Ai the Airy function, namely the non trivial solution of $y''(x) = xy(x)$ going to zero at $\pm\infty$, illustrated on Figure 2. Check that

$$x \longmapsto \mathrm{Ai}\left(\pm\sqrt{2} E^{1/6} \frac{x - x_\pm}{h^{2/3}} \right)$$

is a solution to (1.6). Explain in which neighborhood of $x_\pm$ this approximation is relevant. Observe that this is consistent with Figure 1.

7. (If time allows:) Discuss possible generalizations to a Schrödinger operator $-\frac{h^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ with a confining potential $V \in C^\infty(\mathbb{R})$.

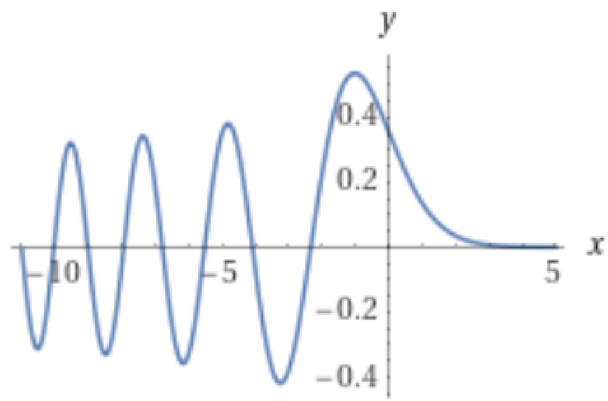

Figure 2: The Airy function.

Exercise 3: Borel summation (optional)

1. Recall why any sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$ are the Taylor coefficients of some smooth function on $\mathbb{R}^d$. (Hint: pick a cut-off function $\chi \in C_c^\infty(\mathbb{R}^d;[0,1])$ equal to 1 near the origin and arrange the formal Taylor series so that, for fixed $x \in \mathbb{R}^d$, only finitely many terms contribute to the sum.)

2. Let $(a_n(h))_{n\in\mathbb{N}}$ be an h-dependent sequence in $C^\infty(\mathbb{R}^d;\mathbb{C})$ and set

$$a(h) := \sum_{n=0}^{\infty} h^n \chi\left(\frac{h}{r_n}\right) a_n(h),$$

where $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers. Convince yourself that for all $N \in \mathbb{N}$, it holds

$$a(h) - \sum_{n=0}^{N} h^n \chi\left(\frac{h}{r_n}\right) a_n(h) = O(h^{N+1})$$

provided $r_n \to 0$ sufficiently fast as $n \to \infty$. (Rigorously, the O notation should refer to a Banach norm or a Fréchet structure in some functional space. The decay of the sequence r_n should take into account the growth of norms of the a_n's.)

2 Semiclassical wave front set and defect measures

We recall the following notation: for some h-dependent quantity $Q(h)$, we write $Q(h) = O(h^\infty)$ if for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that $|Q(h)| \leq C_N h^N, \forall h \in (0,1]$. Recall the definition of the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) := (2\pi h)^{-d/2} \int_{\mathbb{R}^d} u(y) e^{-\frac{i}{h}\xi . y} dy, \qquad \mathcal{F}_h^{-1} u(x) := (2\pi h)^{-d/2} \int_{\mathbb{R}^d} u(\xi) e^{\frac{i}{h}\xi . x} d\xi. \tag{2.1}$$

Also recall the result below from Lecture 2.

Proposition 1. A point (x_0, ξ_0) is not in $\mathrm{WF}_h u$ if and only if there exists a cut-off function ϕ such that

$$\mathcal{F}_h(\phi u) = O(h^\infty) \qquad \text{in a neighborhood of } \xi_0.$$

Exercise 4

As an application of the above proposition, compute $\mathrm{WF}_h e^{\frac{i}{h}\alpha\cdot x}$ for $\alpha \in \mathbb{R}^d$ independent of h, and $\mathrm{WF}_h a$ where $a \in C^\infty(\mathbb{R}^d;\mathbb{C})$ is independent of h.

Exercise 5

Let $\phi \in C^\infty(\mathbb{R}^d;\mathbb{C})$ have no critical points. Compute $\mathrm{WF}_h e^{\frac{i}{h}\phi(x)}$. (Hint: prove it directly, or alternatively, use that the wave front set is coordinate invariant as a subset of $T^*\mathbb{R}^d$ and reduce to the case of a linear phase as in the previous exercise.)

Exercise 6: Computation of defect measures

Compute the defect measures associated with the following families of functions:

1. (bump function going to infinity)
$$u_h(x) := \phi\left(x - \frac{1}{h}\right),$$
where $\phi \in L^2(\mathbb{R})$;

2. (Gaussian wave packet)
$$u_h(x) := (\pi h)^{-d/4} \exp\left(-\frac{|x-x_0|^2}{2h}\right) e^{\frac{i}{h}\xi_0\cdot x}, \qquad x \in \mathbb{R}^d,$$
where (x_0, ξ_0) is a fixed phase space point;

3. (plane wave)
$$u_h(x) := e^{\frac{i}{h^\beta}\alpha\cdot x}, \qquad \alpha \in \mathbb{R}^d,$$
according to the values of $\beta \in \mathbb{R}$.

Now recall the following result from Lecture 2.

Proposition 2. Let $P = \mathrm{Op}_h(p)$ be an h-pseudo-differential operator acting on $\mathbb{R}^d$. If $(u_h)_{h\in(0,1]}$ is a bounded sequence in $L^2(\mathbb{R}^d)$ satisfying $Pu_h = o_{L^2}(1)$ as $h \to 0$, then any semiclassical defect measure μ associated with $(u_h)_h$ satisfies
$$\operatorname{supp} \mu \subset \{p = 0\}.$$
If moreover P is self-adjoint and $Pu_h = o_{L^2}(h)$, then μ is invariant by the Hamiltonian flow of p, namely $\mathsf{H}_p\mu = 0$ where
$$\mathsf{H}_p = \frac{\partial p}{\partial \xi}\cdot\partial_x - \frac{\partial p}{\partial x}\cdot\partial_\xi.$$

Let us also recall the stationnary phase lemma (see Melissa Tacy's lecture.)

Proposition 3 (Stationnary phase)**.** Let $a \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$ and $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$. Suppose ϕ has a unique critical point x_0 in $\operatorname{supp} a$, namely $\partial\phi(x_0) = 0$, and that $\det \partial^2\phi(x_0) \neq 0$. Then it holds
$$(2\pi h)^{-d/2}\int_{\mathbb{R}^d} a(x) e^{\frac{i}{h}\phi(x)}dx \overset{h\to 0}{\sim} \frac{e^{i\frac{\pi}{4}\operatorname{sgn}\partial^2\phi(x_0)}}{|\det \partial^2\phi(x_0)|^{1/2}} a(x_0) e^{\frac{i}{h}\phi(x_0)},$$
where $\operatorname{sgn}\partial^2\phi(x_0)$ is the signature of the Hessian matrix $\partial^2\phi(x_0)$, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

Exercise 7: Lagrangian state

Let $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ have no critical point and $b \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$ be independent of h. Let μ be a semiclassical defect measures associated with
$$u_h(x) := b(x) e^{\frac{i}{h}\phi(x)}.$$

1. Using Proposition 2, quickly recall why $\operatorname{supp}\mu \subset \operatorname{Graph}(d\phi)$. (Hint: introduce the operator $P = \frac{h}{i}\partial_j - \partial_j\phi$.)

2. Now we want to prove the more precise result

$$\mu = |b(x)|^2 \delta_{\{\xi = d\phi(x)\}}. \tag{2.2}$$

Let $a \in C_c^\infty(\mathbb{R}^{2d}; \mathbb{C})$, and denote by $a(x, hD)$ the standard quantization (or left quantization) of a, namely the operator

$$a(x, hD) := \mathcal{F}_h^{-1} \left(a(x, \bullet) \mathcal{F}_h u(\bullet) \right),$$

where $\mathcal{F}_h$ is the semiclassical Fourier transform (2.1).

   (a) Justify why semiclassical defect measures do not depend of the choice of quantization (standard, Weyl...). (Hint: recall how a relates to the Weyl symbol of the operator $a(x, hD)$ for instance.)

   (b) Put the above inner product under the form

   $$(u_h, a(x, hD) u_h)_{L^2} = (2\pi h)^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} a(x, \xi) e^{\frac{i}{h} \Phi_x(y, \xi)} \overline{b(x)} b(y) dy d\xi \right) dx,$$

   with a phase Φ_x to be expressed explicitly.

   (c) We fix $x \in \mathbb{R}^d$ and study the inner integral over (y, ξ) using the stationnary phase asymptotics. Compute the unique critical point (y_0, ξ_0) of Φ_x. Compute the Hessian matrix $\partial^2 \Phi_x$ and check that $|\det \partial^2 \Phi_x(y_0, \xi_0)| = 1$ and $\operatorname{sgn} \partial^2 \Phi_x(y_0, \xi_0) = 0$. (Hint: you can bypass the computation of the signature and check in the end that the exponential factor involving the signature in Proposition 3 has to be equal to 1 since μ is a non-negative Radon measure.)

   (d) Apply Proposition 3 to deduce the sought result (2.2).

Exercise 8: Back to the 1D harmonic oscillator.

Let $(u_h)_{h \in (0,1]}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ such that (1.3) holds with $E_h = 1 + o(h)$. Compute the (unique) semiclassical defect measure associated with this sequence. Compare with the result of Question 3 of Exercise 2.