Introduction to scattering theory

Maxime INGREMEAU (lecture)   Zoïs MOITIER (exercises)

Semiclapp: Semiclassical analysis and applications
13$^{\text{th}}$ — 17$^{\text{th}}$ May 2024

1 What is scattering theory?

In mathematical physics, scattering can refer to several concepts:

1. The long time behaviour of a system of interacting particles (which will typically come from far away, interact, possibly create new particles, which will go far away).
2. The long time behaviour of a nonlinear wave equation, and how it is related to a linear wave equation with constant coefficients.
3. The long time behaviour of a linear wave equation with non-constant coefficients, and how it is related to a linear wave equation with constant coefficients.
4. The spectral theory of an operator with non-constant coefficients (description of its eigenfunctions and resolvent), and how it is related to the spectral theory of a model operator with constant coefficients.

In quantum physics, particles are represented by waves, which can interact either linearly or non-linearly, so that point Item 1 is the physical counterpart of Items 2 and 3. The long time behaviour of a linear evolution problem is always strongly related to the spectral properties of its generator, so that points Items 3 and 4 are strongly related. Historically, scattering theory for linear PDEs (with non-constant coefficients) was more focused on Item 3 (see [3]), but point Item 4 corresponds to the modern point of view (as in [4]), and this is the one we will follow here. In these notes, we will follow the recent book [2], and in particular, part of chapters 2, 3 and 4.

> In all the sequel, we will consider $d = 3$, or $d = 1$ (when it makes proofs easier).

2 The free resolvent

Before describing the resolvent of operators with non-constant coefficients, we need to describe the resolvent of the model operator with constant coefficients, namely, $-\Delta$.

2.1 Definition and meromorphic continuation

For $\Im\lambda > 0$, consider the holomorphic family of operators

$$R_0(\lambda) \coloneqq \left(-\Delta - \lambda^2\right)^{-1} \colon \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d).$$

As an easy application of the spectral theorem, we have

$$\|R_0(\lambda)\|_{\mathrm{L}^2\to\mathrm{L}^2} \le \frac{1}{|\lambda|\Im\lambda}.$$

Proposition 2.1 (Green's kernel of the free resolvent)**.** *For any $f \in \mathrm{L}^2(\mathbb{R}^d)$, and any λ with $\Im\lambda > 0$, we have*

$$\big(R_0(\lambda)f\big)(x) = \int_{\mathbb{R}^d} R_0(x,y;\lambda) f(y)\,\mathrm{d}y, \tag{1}$$

where

$$R_0(x,y;\lambda) = \begin{cases} \dfrac{\mathsf{i}}{2\lambda}\mathsf{e}^{\mathsf{i}\lambda|x-y|}, & \text{if } d=1,\\[2ex] \dfrac{\mathsf{e}^{\mathsf{i}\lambda|x-y|}}{4\pi|x-y|}, & \text{if } d=3. \end{cases}$$

The map $\lambda \mapsto R_0(x,y;\lambda)$ is meromorphic in all $\mathbb{C}$, smooth in x and y, but it decays when $|x-y| \to \infty$ only when $\Im\lambda > 0$. Therefore, for any $\rho \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, the map

$$\rho R_0(\lambda)\rho \colon \mathrm{L}^2(\mathbb{R}^d) \longrightarrow \mathrm{H}^2(\mathbb{R}^d),$$
$$\big(\rho R_0(\lambda)\rho f\big)(x) = \int_{\mathbb{R}^3} \rho(x) R_0(x,y;\lambda)\rho(y) f(y)\,\mathrm{d}y,$$

is well-defined for all $\lambda \in \mathbb{C}$. In other words, the map $R_0(\lambda) \colon \mathrm{L}^2_{\mathrm{comp}}(\mathbb{R}^d) \to \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$, initially defined for $\Im\lambda > 0$, can be meromorphically continued to $\mathbb{C}$ (with a pole only at $\lambda = 0$, when $d = 1$).

Remark 2.2. *When $d = 2$, the Greens's kernel can be expressed in terms of a Hankel function: $R_0(x,y;\lambda) = \frac{\mathsf{i}}{4}\mathsf{H}_0^{(1)}(\lambda|x-y|)$. This function is more complicated than when $d = 1, 3$, and it has a logarithmic singularity at $\lambda = 0$, so that it cannot be continued meromorphically to $\mathbb{C}$. However, most of the results presented here still hold when $d = 2$.*

Remark 2.3. *Here, we defined $R_0(\lambda)$ for $\Im\lambda > 0$, and we extended it to $\Im\lambda \le 0$. We could also have started by defining $R_0(\lambda)$ for $\Im\lambda < 0$, and then extend it to $\Im\lambda \ge 0$. The two procedures don't give the same result!*

The first procedure gives what is called the outgoing *resolvent, sometimes denoted by $R_0(\lambda + \mathsf{i}0)$, to recall that it was first defined for $\Im\lambda > 0$, while the second one is called the* incoming *resolvent, and is sometimes denoted by $R_0(\lambda - \mathsf{i}0)$. This is explained in more details in the following remark.*

Remark 2.4 (Why is it called an outgoing resolvent?)**.** *If $x \in \mathbb{R}$, we may write*

$$\frac{1}{x-\lambda^2} = \mathsf{i}\int_0^{+\infty} \mathsf{e}^{\mathsf{i}(\lambda^2-x)t}\,\mathrm{d}t,$$
$$= -\mathsf{i}\int_{-\infty}^0 \mathsf{e}^{\mathsf{i}(\lambda^2-x)t}\,\mathrm{d}t,$$

but the first expression only makes sense when $\Im(\lambda^2) > 0$, while the second only makes sense when $\Im(\lambda^2) < 0$. By the spectral theorem, we may replace x with $-\Delta$, to obtain

$$\begin{aligned}\left(-\Delta - \lambda^2\right)^{-1} &= \mathrm{i}\int_0^{+\infty} \mathrm{e}^{\mathrm{i}(\lambda^2+\Delta)t}\,\mathrm{d}t, && \textit{if } \Im(\lambda^2) > 0,\\ &= -\mathrm{i}\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}(\lambda^2+\Delta)t}\,\mathrm{d}t, && \textit{if } \Im(\lambda^2) < 0.\end{aligned}$$

Hence, if $\lambda_0 > 0$, there are two reasonable ways of defining $(-\Delta - \lambda_0^2)^{-1}$,

$$\textit{as } \lim_{\varepsilon\searrow 0} \mathrm{i}\int_0^{+\infty} \mathrm{e}^{\mathrm{i}((\lambda_0+\mathrm{i}\varepsilon)^2+\Delta)t}\,\mathrm{d}t \quad \textit{or} \quad \textit{as } -\lim_{\varepsilon\searrow 0} \mathrm{i}\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}((\lambda_0-\mathrm{i}\varepsilon)^2+\Delta)t}\,\mathrm{d}t.$$

The first limit involves $\mathrm{e}^{\mathrm{i}t\Delta}$, which is the Schrödinger propagator at positive times. Hence, waves propagate in the future, from a compact region towards infinity: they are thus outgoing. The second expression involves the Schrödinger propagator at negative times. Here, waves go to infinity in the past: they are thus incoming.

The discussion here use the Schrödinger equation, but we could do a similar argument using the wave propagator.

2.2 Characterization of outgoing functions

If $\Im\lambda > 0$, then the operator $R_0(\lambda)$ is the inverse of $(-\Delta - \lambda^2)\colon \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$. When $\Im\lambda \leq 0$, $R_0(\lambda)$ is only a right-inverse of $(-\Delta - \lambda^2)$:

$$\forall f \in \mathrm{L}^2_{\mathrm{comp}}(\mathbb{R}^d), \quad (-\Delta - \lambda^2)R_0(\lambda)f = f. \tag{2}$$

However, we generally don't have $R_0(\lambda)[(-\Delta - \lambda^2)u] = u$. This will hold only if u belongs to the image of $R_0(\lambda)$, which we now describe.

Proposition 2.5. *Let $u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $(-\Delta - \lambda^2)u = 0$ outside a compact set K. The following conditions are equivalent:*

1. *There exists $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $u = R_0(\lambda)f$.*
2. *u satisfies the* **Sommerfeld radiation conditions***:*

$$\frac{\partial u(x)}{\partial |x|} - \mathrm{i}\lambda u(x) = o_{|x|\to+\infty}\left(\frac{1}{|x|^{(d-1)/2}}\right). \tag{3}$$

3. *u satisfies*

$$\frac{\partial u(x)}{\partial |x|} - \mathrm{i}\lambda u(x) = \mathcal{O}_{|x|\to+\infty}\left(\frac{1}{|x|^{(d+1)/2}}\right).$$

4. *There exists a function $h\colon \mathbb{S}^{d-1} \to \mathbb{C}$ such that*

$$u(|x|\omega) = \frac{\mathrm{e}^{\mathrm{i}\lambda|x|}}{|x|^{(d-1)/2}}h(\omega) + \mathcal{O}_{|x|\to+\infty}\left(\frac{1}{|x|^{(d+1)/2}}\right).$$

If one of these conditions is satisfied, we say that f is **outgoing**.

3 The resolvent for a perturbation of $-\Delta$

Now, we consider an operator

$$P = -\Delta + V, \tag{4}$$

with $V \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbb{R}^d)$. More generally, all the results presented here would also apply to

$$Pu = -\nabla(A\nabla u) + Vu, \tag{5}$$

where $A(x)$ is a positive definite matrix, with $A \equiv \mathrm{Id}$ outside a compact set.

Lemma 3.1. *There exists $C(V) > 0$ such that, for all $\lambda \in \mathbb{C}$ with $\Im\lambda > C(V)$, the operator $(-\Delta + V - \lambda^2)$ is invertible.*

Proof. First, note that for all $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$, we have

$$(-\Delta + V - \lambda^2)R_0(\lambda) = \mathrm{Id} + VR_0(\lambda). \tag{6}$$

Multiplication by V is a bounded operator, and R_0 is small if $\Im\lambda$ is large enough. Therefore, $\|VR_0(\lambda)\|_{\mathrm{L}^2\to\mathrm{L}^2} < 1$ for $\Im\lambda$ large enough. We may hence invert $\mathrm{Id}+VR_0(\lambda)$ by a Neumann series for $\Im\lambda \gg 1$. The result follows. □

Theorem 3.2. *Let $V \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^3)$. The family of operators*

$$(-\Delta + V - \lambda^2)^{-1} \colon \mathrm{L}^2_{\mathrm{comp}}(\mathbb{R}^d) \longrightarrow \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$$

extends as a meromorphic family of operators to $\lambda \in \mathbb{C}$. Its poles are called the (scattering) resonances *of $-\Delta + V$. If $\lambda \in \mathbb{C}$ is a resonance, then* $\ker(-\Delta + V - \lambda^2)$ *is finite dimensional.*

Idea behind the proof. Write

$$\big(-\Delta + V - \lambda^2\big)R_0(\lambda) = \mathrm{Id} + VR_0(\lambda),$$

so that, formally,

$$\big(-\Delta + V - \lambda^2\big)^{-1} = R_0(\lambda)\big(\mathrm{Id} + VR_0(\lambda)\big)^{-1} \colon \mathrm{L}^2_{\mathrm{comp}} \longrightarrow \mathrm{H}^2_{\mathrm{loc}}. \tag{7}$$

Using analytic Fredholm theory, one can show that $(\mathrm{Id} + VR_0(\lambda))^{-1}$ is a meromorphic family of operators. □

In particular, we see from (7) that

- $\lambda \in \mathbb{C} \setminus \{0\}$ is a scattering resonance if and only if there exists a non-trivial $u \in \mathrm{L}^2_{\mathrm{comp}}$ such that $u = -VR_0(\lambda)u$. Writing $w = R_0(\lambda)u$ and applying $R_0(\lambda)$ to the equation, we also get the existence of $w \in \mathrm{H}^2_{\mathrm{loc}}$ such that $w = -R_0(\lambda)Vw$.
- If λ is not a resonance, then $(-\Delta + V - \lambda^2)^{-1}$ and $R_0(\lambda)$ have the same range, which can be described using Proposition 2.5.
- There can be only finitely many resonances in $\{\Im z > 0\}$, of the form $\mathrm{i}y$, where $-y^2$ is a negative eigenvalue of P.

Theorem 3.3. *An operator of the form (4) or (5) has no resonance in $\mathbb{R}$.*

In other words, for all $\lambda \in \mathbb{R}$, and all $f \in \mathrm{L}^2_{\mathrm{comp}}(\mathbb{R}^d)$, there exists a unique u such that

$$\begin{cases} -\Delta u + Vu - \lambda^2 u = f \\ \dfrac{\partial u(x)}{\partial |x|} - \mathrm{i}\lambda u(x) = o_{x\to+\infty}\left(\dfrac{1}{|x|^{(d-1)/2}}\right) \end{cases} \tag{8}$$

Proof when $d = 1$. Let $[-K,\ K]$ contain the support of V, let $u = -VR_0(\lambda)u$, and let $w = R_0(\lambda)u$. In particular,

$$(P - \lambda^2)w = Vw = -(-\Delta - \lambda^2)w,$$

and this function vanishes outside $[-K,\ K]$. Furthermore, outside $[-K,\ K]$, w must be of the form $w(x) = a_\pm \mathrm{e}^{\pm\mathrm{i}\lambda x}$ for some $a_\pm \in \mathbb{C}$. Now, since P is self-adjoint, we have

$$\begin{aligned} 0 &= \Im\big(\langle w, (P - \lambda^2)w\rangle\big), \\ &= \Im\left(\int_{-K}^{K} w(-\Delta - \lambda^2)\overline{w}\,\mathrm{d}x\right), \\ &= \Im\big([-w\overline{w}']_{-K}^{K}\big), \\ &= \lambda\big(|a_+|^2 + |a_-|^2\big). \end{aligned}$$

Therefore, when must have $a_+ = a_- = 0$, and, by the (linear) Cauchy-Lipschitz theorem, we must have $w = 0$. □

When $d = 3$, the argument is similar: we do an integration by parts to show that $w \equiv 0$ outside a compact set. However, to conclude that $w = 0$, one cannot use the Cauchy-Lipschitz theorem, and one must use a ***unique continuation principle*** (which is much more complicated).

4 Perfectly matched layers and Complex scaling

Both in theoretical and numerical considerations, when considering the problem (8), it is unpleasant to consider a PDE with a boundary condition at infinity.

The method of Perfectly Matched Layers (*a.k.a.* Complex Scaling) allows adding non-self adjoint terms in the equation $P - \lambda^2 u = f$ which will account for the Sommerfeld radiation condition (3).

4.1 Perfectly matched layers when $d = 1$

Let $g \in \mathscr{C}^1(\mathbb{R};\mathbb{R})$ with $g = 0$ on $[-R,\ R]$, with $g' \geq 0$, and $g(x) = x$ when $|x| \geq 2R$. We define an operator $P_g \colon \mathrm{H}^2 \to \mathrm{L}^2$ by

$$P_g u := -\frac{1}{1 + \mathrm{i}g'}\left(\frac{1}{1 + \mathrm{i}g'}u'\right)'. \tag{9}$$

Proposition 4.1. *For any $\lambda \in \mathbb{R}$, $P_g + V - \lambda^2$ is invertible. If $f \in \mathrm{L}^2_{\mathrm{comp}}(\mathbb{R})$ with $\mathrm{supp}(f) \subset [-R,\, R]$, the solution u of $(P_g + V - \lambda^2)u = f$ decays exponentially at infinity. Furthermore, if v is the solution of $-v'' + Vv - \lambda^2 v = f$ satisfying the Sommerfeld radiation condition (3), then we have $v \equiv u$ in $[-R,\, R]$.*

Hence, Proposition 4.1 allows us to solve $(P_g + V - \lambda^2)u = f$ instead of (8), which is much easier, since we do not have conditions at infinity any more. However, proving Proposition 4.1 directly (without using the complex scaling introduced below) is hard (try to do it!), and we don't really understand where the operator P_g comes from.

4.2 Another point of view: complex scaling

We will now explain the proof of Proposition 4.1 when $V \equiv 0$, by the method of *complex scaling*. The case $V \neq 0$ can be recovered by using (7).

We introduce the curve $\gamma(t) = t + \mathrm{i}g(t) \colon \mathbb{R} \to \Gamma \subset \mathbb{C}$. For $u \colon \mathbb{R} \to \mathbb{C}$ a function on $\mathbb{R}$, we define a function $u_\Gamma \coloneqq u \circ \gamma^{-1} \colon \Gamma \to \mathbb{C}$. For $v \colon \Gamma \to \mathbb{C}$, we define the operator[1]

$$\partial_\Gamma v \coloneqq \frac{1}{\gamma'(t)} (v \circ \gamma)', \tag{10}$$

so that $\partial_\Gamma u_\Gamma = \partial u$, and $P_g u = -\partial_\Gamma^2 u_\Gamma$. If v is the unique solution of $-v'' - \lambda^2 v = f$ which satisfies the Sommerfeld radiation condition (3), we have $v(x) = a_\pm \mathrm{e}^{\pm \mathrm{i}\lambda x}$ when $\pm x \geq R$. We then define a new function $w \colon \Gamma \to \mathbb{C}$ as follows:

$$w(z) = \begin{cases} a_+ \mathrm{e}^{\mathrm{i}\lambda z}, & \text{if } \Re z > R, \\ v(z) = v(x), & \text{if } \Re z \in [-R,\, R], \\ a_- \mathrm{e}^{-\mathrm{i}\lambda z}, & \text{if } \Re z < -R. \end{cases}$$

A direct computation shows that $(-\partial_\Gamma^2 - \lambda^2)w = f_\Gamma$. Therefore, if we write $u \coloneqq w \circ \gamma \colon \mathbb{R} \to \mathbb{C}$, we have $(P_g - \lambda^2)u = f$. This will give us the existence of a solution, if we check that $u \in \mathrm{H}^2(\mathbb{R})$. The function u is clearly in $\mathrm{H}^2_{\mathrm{loc}}$, and the fact that[2] $\pm \Im \gamma(x) > (|x| - R)$ when $\pm x > R$ implies that it decays exponentially at infinity. Note that u coincides with v on $[-R,\, R]$, as announced. The uniqueness is left as an exercise.

4.3 Perfectly matched layers in higher dimension

When $d = 3$, an analogue of (9) can be constructed as follows. We still consider $g \in \mathscr{C}^1(\mathbb{R}; \mathbb{R})$ with $g = 0$ on $[-R,\, R]$, with $g' \geq 0$, and $g(x) = x$ when $|x| \geq 2R$. We define an operator $P_g \colon \mathrm{H}^2 \to \mathrm{L}^2$ by

$$P_g \coloneqq \big((\mathrm{Id} + \mathrm{i}M_g)^{-1}\nabla\big)^2,$$

where

$$M_g(x) \coloneqq \frac{g(|x|)}{|x|^3}\big(|x|^2\mathrm{Id} - x \otimes x\big) + \frac{g'(|x|)}{|x|^2} x \otimes x,$$

with $x \otimes x$ is the orthogonal projection on x. The analogue of Proposition 4.1 then holds.

[1] One can check that the operator ∂_Γ does not depend on the choice of the parametrization γ.
[2] Note that this the place where we use the Sommerfeld radiation condition.

References

[1] T. Christiansen. "Schrödinger operators with complex-valued potentials and no resonances". In: *Duke Mathematical Journal* 133.2 (2006). DOI: 10.1215/s0012-7094-06-13324-0.

[2] S. Dyatlov and M. Zworski. *Mathematical Theory of Scattering Resonances*. Vol. 200. American Mathematical Society, 2019. DOI: 10.1090/gsm/200.

[3] P. D. Lax and R. S. Phillips. *Scattering theory*. Rev. ed. Vol. 26. Boston etc.: Academic Press, Inc., 1989. URL: https://zbmath.org/0697.35004.

[4] R. B. Melrose. *Geometric scattering theory*. Cambridge University Press, 1995. URL: https://zbmath.org/0849.58071.

5 Exercises

Exercise 1.
Prove Proposition 2.1 when $d = 1$.

Hint: Young's convolution inequality.

Exercise 2. Hard exercise
The aim of this exercise is to prove Proposition 2.1 when $d = 3$.

Question 1. Show that $R_0(x, y; \lambda) = G_\lambda(x - y)$, where

$$G_\lambda = \frac{1}{(2\pi)^{3/2}} \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - \lambda^2}\right).$$

Recall the Fourier transform and its inverse

$$\mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \, \mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}[g](x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}} g(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \, \mathrm{d}\xi.$$

Question 2. Show that

$$\int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}r\omega\cdot x} \, \mathrm{d}\omega = \frac{2\pi}{\mathrm{i}r|x|}\left(\mathrm{e}^{\mathrm{i}r|x|} - \mathrm{e}^{-\mathrm{i}r|x|}\right). \tag{11}$$

Question 3. Deduce from the previous two questions that

$$G_\lambda(x) = \frac{1}{8\mathrm{i}\pi^2|x|} \int_{\mathbb{R}} \frac{r}{r^2 - \lambda^2} \mathrm{e}^{\mathrm{i}r|x|} \, \mathrm{d}r - \frac{1}{8\mathrm{i}\pi^2|x|} \int_{\mathbb{R}} \frac{r}{r^2 - \lambda^2} \mathrm{e}^{-\mathrm{i}r|x|} \, \mathrm{d}r$$

Question 4. Apply the residue theorem to conclude.

Hint: $\sin\theta \leq \frac{2}{\pi}\theta$ *for* $0 \leq \theta \leq \frac{\pi}{2}$.

Solution 2.
Question 1. We have $f = (-\Delta - \lambda^2) R_0(\lambda) f$, so applying the Fourier transform, we obtain

$$\mathcal{F}[f](\xi) = \left(|\xi|^2 - \lambda^2\right) \mathcal{F}[R_0(\lambda) f](\xi),$$

and we deduce

$$R_0(\lambda) f = \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2 - \lambda^2} \mathcal{F}[f]\right] = \frac{1}{(2\pi)^{3/2}} \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2 - \lambda^2}\right] \star f.$$

Therefore, (1) holds, with $R_0(x, y; \lambda) = G_\lambda(x - y)$, where

$$G_\lambda(x) = \frac{1}{(2\pi)^{3/2}} \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2 - \lambda^2}\right](x).$$

Question 2. First, observe that the function $x \mapsto \int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}r\omega\cdot x} \, \mathrm{d}\omega$ is invariant by rotation so its value at x and $(0, 0, |x|)$ is the same. We may therefore assume that $x = (0, 0, |x|)$ in the canonical basis of $\mathbb{R}^3$. Working in spherical coordinates, we obtain

$$\begin{aligned}
\int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}r\omega\cdot x} \, \mathrm{d}\omega &= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \mathrm{e}^{\mathrm{i}r|x|\cos\varphi} \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta \\
&= 2\pi \left[-\frac{\mathrm{e}^{\mathrm{i}r|x|\cos\varphi}}{\mathrm{i}r|x|}\right]_{\varphi=0}^{\pi} \\
&= \frac{2\pi}{\mathrm{i}r|x|}\left(\mathrm{e}^{\mathrm{i}r|x|} - \mathrm{e}^{-\mathrm{i}r|x|}\right).
\end{aligned}$$

Question 3. We compute

$$\mathcal{F}^{-1}\left[\frac{1}{|\xi|^2-\lambda^2}\right](x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{\mathrm{i}x\cdot\xi}}{|\xi|^2-\lambda^2}\,\mathrm{d}x = \frac{1}{(2\pi)^{3/2}}\int_0^{+\infty}\int_{\mathbb{S}^2}\frac{\mathrm{e}^{\mathrm{i}r\,x\cdot\omega}}{r^2-\lambda^2}r^2\,\mathrm{d}r\,\mathrm{d}\omega,$$

and using (11), we obtain

$$\begin{aligned} G_\lambda(x) &= \frac{1}{(2\pi)^3}\int_0^{+\infty}\int_{\mathbb{S}^2}\frac{\mathrm{e}^{\mathrm{i}r\,x\cdot\omega}}{r^2-\lambda^2}r^2\,\mathrm{d}r\,\mathrm{d}\omega, \\ &= \frac{1}{(2\pi)^2\mathrm{i}|x|}\int_0^{+\infty}\frac{r}{r^2-\lambda^2}\left(\mathrm{e}^{\mathrm{i}r|x|}-\mathrm{e}^{-\mathrm{i}r|x|}\right)\mathrm{d}r \\ &= \frac{1}{8\mathrm{i}\pi^2|x|}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\left(\mathrm{e}^{\mathrm{i}r|x|}-\mathrm{e}^{-\mathrm{i}r|x|}\right)\mathrm{d}r \\ &= \frac{1}{8\mathrm{i}\pi^2|x|}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\mathrm{e}^{\mathrm{i}r|x|}\,\mathrm{d}r - \frac{1}{8\mathrm{i}\pi^2|x|}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\mathrm{e}^{-\mathrm{i}r|x|}\,\mathrm{d}r \end{aligned}$$

Question 4. Now, the map $r \mapsto \frac{r}{r^2-\lambda^2}\mathrm{e}^{\mathrm{i}r|x|}$ is meromorphic, with simple poles at $r=\pm\lambda$. It goes to zero when $r\to+\infty$ with $\Im r\geq 0$, so we can use the residue theorem on the upper half disc of centre 0 and radius $R\to+\infty$ to obtain

$$\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\mathrm{e}^{\mathrm{i}r|x|}\,\mathrm{d}r = 2\mathrm{i}\pi\,\mathrm{Res}_{r=\lambda}\left(\frac{r}{r^2-\lambda^2}\mathrm{e}^{\mathrm{i}r|x|}\right) = 2\mathrm{i}\pi\frac{\lambda}{2\lambda}\mathrm{e}^{\mathrm{i}\lambda|x|} = \mathrm{i}\pi\mathrm{e}^{\mathrm{i}\lambda|x|}$$

Similarly, using the residue theorem on the lower half disc of centre 0 and radius $R\to+\infty$, we have

$$\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\mathrm{e}^{-\mathrm{i}r|x|}\,\mathrm{d}r = -2\mathrm{i}\pi\frac{(-\lambda)}{-2\lambda}\mathrm{e}^{\mathrm{i}\lambda|x|} = -\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\lambda|x|},$$

and finally we get

$$G_\lambda(x) = \frac{1}{8\mathrm{i}\pi^2|x|}2\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\lambda|x|} = \frac{\mathrm{e}^{\mathrm{i}\lambda|x|}}{4\pi|x|},$$

which proves the result.

Exercise 3.
Let $V\in\mathrm{L}^\infty(\mathbb{R})$ be a real-valued potential and $\lambda\in\mathbb{R}\setminus\{0\}$. Assume that we have a function $u\in\mathrm{H}^2_{\mathrm{loc}}(\mathbb{R})$ such that $-u''+Vu=\lambda^2u$.

Question 1. Show that there exists $R>0$ and $\alpha_\pm,\beta_\pm\in\mathbb{C}$ such that

$$u(x) = \alpha_\pm\mathrm{e}^{\mathrm{i}\lambda|x|}+\beta_\pm\mathrm{e}^{-\mathrm{i}\lambda|x|}, \qquad \text{for } \pm x\geq R.$$

Question 2. Show that $|\alpha_-|^2+|\alpha_+|^2=|\beta_-|^2+|\beta_+|^2$.

Hint: Wronskian.

Exercise 4.
Prove Proposition 2.5 when $d=1$.

Exercise 5.

Let $f \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^3)$. Show that, for any $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$, we have

$$\Big(\big(-\Delta - \lambda^2\big)^{-1} f\Big)(r\omega) = \frac{\mathrm{e}^{\mathrm{i}\lambda r}}{4\pi r}\,\mathcal{F}[f](-\lambda\omega) + \mathcal{O}_{r\to+\infty}\Big(\frac{1}{r^2}\Big).$$

This exercise proves part of Proposition 2.5.

Exercise 6.

Let $V_0 \in \mathbb{R}$ and define the step potential $V(x) = V_0\,\mathbf{1}_{|x|\le 1}(x)$.

Question 1. Derive a necessary and sufficient criterion for $\lambda \in \mathbb{C}$ to be a resonance of the equation $-\partial_x^2 + V$.

Question 2. Find the asymptotic expansion of the resonances as $\Re\lambda \to +\infty$.

Question 3. Find a condition on V_0 to have negative eigenvalues (*a.k.a.* bound state).

Exercise 7. Symmetry of resonances for real-valued potentials

For a real-valued potential $V \in \mathrm{L}^\infty(\mathbb{R}^d)$ in dimension 1 and 3, show that if λ is a resonance then $-\overline{\lambda}$ is also a resonance.

Hint: Find a link between $R_0(\lambda)$ and $R_0(-\overline{\lambda})$.

Exercise 8. This exercise is an adaptation of [1].

In $\mathbb{R}^3$, consider cylindrical coordinates $(r,\theta,z) \in [0,\,\infty)\times[0,\,2\pi)\times\mathbb{R}$. Recall that in such coordinates, the Laplacian takes the form

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Consider the potential $V \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbb{R}^3;\mathbb{C})$ given by

$$V(r,\theta,z) := \mathrm{e}^{\mathrm{i}\theta}\,\mathbf{1}_{r\le 1}(r)\,\mathbf{1}_{|z|\le 1}(z).$$

The aim of this exercise is to show that $-\Delta + V$ has no resonances.

If $\ell \in \mathbb{Z}$, we denote by Π_ℓ the projection of the ℓ^{th} Fourier mode:

$$\Pi_\ell u(r,\theta,z) = \frac{\mathrm{e}^{\mathrm{i}\ell\theta}}{2\pi}\int_0^{2\pi} u(r,\phi,z)\mathrm{e}^{-\mathrm{i}\ell\phi}\,\mathrm{d}\phi.$$

Question 1. Let $R > 0$. Show that there exists $C(R) > 0$ such that for all $\ell \in \mathbb{Z}$ and all $u \in \mathrm{H}^2(\mathbb{R}^3)$ supported in $B(0,R)$ and which satisfies $\Pi_\ell u = u$, we have

$$\langle -\Delta u, u\rangle \ge C\ell^2\|u\|^2_{\mathrm{L}^2}.$$

Question 2. Let $\rho \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^3)$ which does not depend on θ. Show that for all $\lambda \in \mathbb{C}$, there exists $C > 0$ depending on ρ and λ such that for all $\ell \in \mathbb{Z}$,

$$\Big\|\Pi_\ell\rho\big(-\Delta - \lambda^2\big)^{-1}\rho\Pi_\ell\Big\|_{\mathrm{L}^2\to\mathrm{L}^2} \le \frac{C(\lambda)}{1+|\ell|}.$$

Question 3. Show that, if u is a resonant state, there exists $C > 0$ such that we have, for all $\ell \in \mathbb{Z}$,

$$\|\Pi_{j+1}u\|_{\mathrm{L}^2} \le \frac{C}{1+|j|}\|\Pi_j u\|_{\mathrm{L}^2}.$$

Question 4. Conclude that $-\Delta + V$ has no resonances in $\mathbb{C}$.

Exercise 9.
Check the uniqueness part in Proposition 4.1, meaning that the problem $(P_g + V - \lambda^2)u = f$ has a unique solution.
Hint: Do the same argument as in the Complex Scaling section, but the other way round.