Semiclapp summer school – Scattering theory
Exercise session I: lower bounds on resonance width in dimension one

The main goal of this session is to show the following Theorem, which is an easy one-dimensional version of a Theorem due to Burq [1, 2].

Theorem 1. *Suppose that* $P(h) := -h^2\Delta + V$, $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$, *and that* $E > 0$. *Then, there exists* $c = c(V, E)$ *such that for* $0 < h < h_0$,

$$\operatorname{Re} z \in [E/2, E],\ z \in \operatorname{Res}(P(h)) \implies |\operatorname{Im} z| > e^{-c/h}.$$

The first part of the following follows [3, §2.8.2]; the second part follows [4, §4.1].

I – Proof of Theorem 1

1. (Preliminary) Suppose that $u \in H^2([-R, R])$ solves

$$((hD_x)^2 + V(x) - z)u(x) = 0, \quad x \in [-R, R], \quad z \in \mathbb{C}.$$

   By integrating $(hD_x)^2 + V(x) - z)u\overline{u} - \overline{(hD_x)^2 + V(x) - z)u}u$, show that

$$\operatorname{Im} z \int_{-R}^{R} |u(x)|^2 dx = -h^2 \Big[\operatorname{Im} \partial_x u \overline{u} \Big]_{-R}^{R}.$$

2. (A one-dimensional Carleman estimate) The goal of this question is to show that, for any $u \in H^2_{\mathrm{comp}}$ and any $M > 0$

$$\|e^{-Mx/h}(hD_x)^2 e^{Mx/h} u\|_{L^2} \geq M^2 \|u\|_{L^2}.$$

   (a) Show that $e^{-Mx/h}(hD_x)^2 e^{Mx/h} u = (hD_x - iM)^2 u$.

   (b) Conclude by using the L^2-isometry property of the semiclassical Fourier transform.

We now suppose that z is a resonance with $\operatorname{Re} z \in [E/2, E]$. Recall that there exists a resonant state u, satisfying

$$((hD_x)^2 + V(x) - z)u = 0, \quad u(x) = A_\pm e^{\pm i\sqrt{z}x/h} \text{ for } \pm x \gg 1, \; A_\pm \neq 0.$$

We assume that $\operatorname{Im} z > -h$ (otherwise there is nothing to prove).

3. Show that for R sufficiently large

$$\left| \left[\operatorname{Im} \partial_x u \overline{u} \right]_{-R}^{R} \right| \geq C \int_{R \leq |x| \leq R+1} |u(x)|^2 dx,$$

   with a constant C depending on R and E.

4. The goal of this question is to show that

$$\int_{-R}^{R} |u(x)|^2 dx \leq C e^{c/h} \int_{R \leq |x| \leq R+1} |u(x)|^2 dx.$$

   (a) Let $M > 0$ so that $M^2 > \|V\|_{L^\infty} + 1$. Show that for any $v \in H^2_{\mathrm{comp}}$

$$\|e^{-Mx/h}((hD_x)^2 + V)e^{Mx/h} v\|_{L^2} \geq \|v\|_{L^2}.$$

   (b) Conclude by applying the above to $v := e^{Mx/h}$ where $\chi \in C_c^\infty((-R-1, R+1); [0,1])$ is equal to one on $[-R, R]$.

5. Conclude.

II – Toward more general Carleman estimates

At the core of the proof of [1, 2] of the lower bounds on resonance width for general operators still lie Carleman estimates – with well chosen exponential weights, and that have to be glued carefully. We now aim to give an idea of more general such Carleman estimates. Let $\Omega \subset \mathbb{R}^d$ be open and P a semiclassical differential operator of order 2, formally self-adjoint and of the form

$$P = \sum_{|\alpha| \leq 2} a_\alpha(x; h)(hD_x)^\alpha, \quad a_\alpha \in C^\infty(\Omega),$$

with

$$a_\alpha(x; h) = a^0_\alpha(x) + O(h) \quad \text{in } C^\infty(\Omega).$$

The semiclassical principal symbol of P writes

$$p(x, \xi) = \sum_{|\alpha| \leq 2} a^0_\alpha(x)\xi^\alpha = a(x, \xi) + \sum_{|\alpha| \leq 1} a^0_\alpha(x)\xi^\alpha,$$

where $a(x,\xi)$ is the ordinary principal symbol of $h^{-2}P$. We assume that P is elliptic in the classical sense:

$$a(x,\xi) \geq \frac{1}{C}|\xi|^2.$$

Let $\phi \in C^\infty(\Omega, \mathbb{R})$, and consider

$$P_\phi := e^{\phi(x)/h} P e^{-\phi(x)/h}.$$

We assume that

$$p_\phi(x,\xi) = 0 \implies \frac{1}{i}\{p_\phi, \overline{p_\phi}\} < 0.$$

We will show the following basic Carleman estimate:

Lemma 2. *Let $W \Subset \Omega$ be open. Then, there exists constants $C > 0, h_0 > 0$ so that for all $0 < h \leq h_0$ and $u \in C_0^\infty(W)$*

$$\|P_\phi u\|_{L^2} \geq \frac{1}{C} h^{1/2} \|u\|_{H_h^2}.$$

6. Show that the semiclassical principal symbol p_ϕ of P_ϕ is given by

$$p_\phi(x,\xi) = p(x, \xi + i\phi'(x)).$$

7. Show that

$$\frac{1}{h}\|P_\phi u\|_{L^2}^2 \geq \langle \frac{1}{h}[P_\phi^*, P_\phi]u, u\rangle.$$

8. Show that there exists $b, C_1 > 0$ so that

$$\frac{1}{i}\{\overline{p_\phi}, p_\phi\} + b|p_\phi(x,\xi)|^2 \geq \frac{1}{C_1}\langle \xi \rangle^4 \quad \text{on } \widetilde{W} \times \mathbb{R}^d,$$

where $\widetilde{W} \Subset \Omega$ with $W \Subset \widetilde{W}$ is open.

9. Deduce from the above that there exists $C_2 > 0$ so that

$$\langle \frac{1}{h}[P_\phi^*, P_\phi]u, u\rangle + b\|P_\phi u\|_{L^2}^2 \geq \frac{1}{C_2}\|u\|_{H_h^2}^2.$$

10. Conclude.

References

[1] Nicolas Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.

[2] Nicolas Burq. Lower bounds for shape resonances widths of long range Schrödinger operators. *Amer. J. Math.*, 124(4):677–735, 2002.

[3] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[4] Johannes Sjöstrand. *Lectures on resonances*.