Semiclapp summer school – Scattering theory
Exercise session I: lower bounds on resonance width
in dimension one

The main goal of this session is to show the following Theorem, which is
an easy one-dimensional version of a Theorem due to Burq [1, 2].

**Theorem 1.** Suppose that $P(h) := -h^2 \Delta + V, V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R})$, and that $E > 0$. Then, there exists $c = c(V, E)$ such that for $0 < h < h_0$,

$$\Re z \in [E/2, E], \quad z \in \text{Res}(P(h)) \implies |\Im z| > e^{-c/h}.$$ 

The first part of the following follows [3, \S 2.8.2]; the second part follows [4, \S 4.1].

**I – Proof of Theorem 1**

1. (Preliminary) Suppose that $u \in H^2([-R, R])$ solves

$$(hD_x)^2 + V(x) - z)u(x) = 0, \quad x \in [-R, R], \quad z \in \mathbb{C}.$$ 

By integrating $(hD_x)^2 + V(x) - z)u \overline{u} - (hD_x)^2 + V(x) - z)uu$, show that

$$\Im z \int_{-R}^{R} |u(x)|^2 \, dx = -h^2 \left[ \Im \partial_x u \overline{u} \right]_{-R}^{R}.$$ 

2. (A one-dimensional Carleman estimate) The goal of this question is to show that, for any $u \in H^2_{\text{comp}}$ and any $M > 0$

$$\|e^{-Mx/h}(hD_x)^2 e^{Mx/h}u\|_{L^2} \geq M^2 \|u\|_{L^2}.$$ 

(a) Show that $e^{-Mx/h}(hD_x)^2 e^{Mx/h}u = (hD_x - iM)^2 u$.

(b) Conclude by using the $L^2$-isometry property of the semiclassical Fourier transform.
We now suppose that $z$ is a resonance with $\text{Re} \, z \in [E/2, E]$. Recall that there exists a resonant state $u$, satisfying

$$((hD_x)^2 + V(x) - z)u = 0, \quad u(x) = A_{\pm} e^{\pm i \sqrt{z} x / h} \quad \text{for} \quad \pm x \gg 1, \quad A_{\pm} \neq 0.$$  

We assume that $\text{Im} \, z > -h$ (otherwise there is nothing to prove).

3. Show that for $R$ sufficiently large,

$$\left| \left[ \text{Im} \partial_x u \bar{u} \right]_{-R}^{R} \right| \geq C \int_{R \leq |x| \leq R+1} |u(x)|^2 dx,$$

with a constant $C$ depending on $R$ and $E$.

4. The goal of this question is to show that

$$\int_{-R}^{R} |u(x)|^2 dx \leq C e^{c/h} \int_{R \leq |x| \leq R+1} |u(x)|^2 dx.$$

(a) Let $M > 0$ so that $M^2 > \| V \|_{L^\infty} + 1$. Show that for any $v \in H^2_{\text{comp}}$

$$\| e^{-Mx/h} ((hD_x)^2 + V) e^{Mx/h} v \|_{L^2} \geq \| v \|_{L^2}.$$  

(b) Conclude by applying the above to $v := e^{Mx/h}$ where $\chi \in C^\infty_c((-R-1, R+1); [0, 1])$ is equal to one on $[-R, R]$.

5. Conclude.

II – Toward more general Carleman estimates

At the core of the proof of [1, 2] of the lower bounds on resonance width for general operators still lie Carleman estimates – with well chosen exponential weights, and that have to be glued carefully. We now aim to give an idea of more general such Carleman estimates. Let $\Omega \subset \mathbb{R}^d$ be open and $P$ a semiclassical differential operator of order 2, formally self-adjoint and of the form

$$P = \sum_{|\alpha| \leq 2} a_{\alpha}(x; h)(hD_x)^\alpha, \quad a_{\alpha} \in C^\infty(\Omega),$$

with

$$a_{\alpha}(x; h) = a_{\alpha}^0(x) + O(h) \quad \text{in} \quad C^\infty(\Omega).$$

The semiclassical principal symbol of $P$ writes

$$p(x, \xi) = \sum_{|\alpha| \leq 2} a_{\alpha}^0(x) \xi^\alpha = a(x, \xi) + \sum_{|\alpha| \leq 1} a_{\alpha}^0(x) \xi^\alpha,$$
where \( a(x, \xi) \) is the ordinary principal symbol of \( h^{-2}P \). We assume that \( P \) is elliptic in the classical sense:

\[
a(x, \xi) \geq \frac{1}{C} |\xi|^2.
\]

Let \( \phi \in C^\infty(\Omega, \mathbb{R}) \), and consider

\[
P_\phi := e^{\phi(x)/h} P e^{-\phi(x)/h}.
\]

We assume that

\[
p_\phi(x, \xi) = 0 \implies \frac{1}{i\hbar} \{ p_\phi, p_\phi \} < 0.
\]

We will show the following basic Carleman estimate:

**Lemma 2.** Let \( W \Subset \Omega \) be open. Then, there exists constants \( C > 0, \hbar_0 > 0 \) so that for all \( 0 < \hbar \leq \hbar_0 \) and \( u \in C_0^\infty(W) \)

\[
\| P_\phi u \|_{L^2} \geq \frac{1}{C} \hbar^{1/2} \| u \|_{H^2_\hbar}.
\]

6. Show that the semiclassical principal symbol \( p_\phi \) of \( P_\phi \) is given by

\[
p_\phi(x, \xi) = p(x, \xi + i\phi'(x)).
\]

7. Show that

\[
\frac{1}{\hbar} \| P_\phi u \|_{L^2}^2 \geq \langle \frac{1}{\hbar} [ P_\phi^*, P_\phi ] u, u \rangle.
\]

8. Show that there exists \( b, C_1 > 0 \) so that

\[
\frac{1}{i\hbar} \{ p_\phi, p_\phi \} + b|p_\phi(x, \xi)|^2 \geq \frac{1}{C_1} (\xi)^4 \quad \text{on} \quad \tilde{W} \times \mathbb{R}^d,
\]

where \( \tilde{W} \Subset \Omega \) with \( W \Subset \tilde{W} \) is open.

9. Deduce from the above that there exists \( C_2 > 0 \) so that

\[
\langle \frac{1}{\hbar} [ P_\phi^*, P_\phi ] u, u \rangle + b\| P_\phi u \|_{L^2}^2 \geq \frac{1}{C_2} \| u \|_{H^2_\hbar}^2.
\]

10. Conclude.
References


