Semiclapp summer school – Scattering theory Exercise session II: lower bounds on resolvent for trapping perturbations

The main goal of this session is to show the following Theorem, due to Bony-Burq-Ramond [1], which gives a *lower bound* on the norm of the resolvent in the case of arbitrary trapping.

In the following, we denote $K_E \subset T^*\mathbb{R}^d$ the trapped set at energy E for the potential V and

$$R(E,h) := (P - E - i0)^{-1}, \quad P := -h^2\Delta + V.$$

Theorem 1. *Suppose that* $E_0 > 0$, $K_{E_0} \neq \emptyset$, *and* $\chi \in C_c^\infty(\mathbb{R}^d)$ *is so that* $\chi = 1$ *near* $\pi(K_{E_0})$. *Then, there exists* $C_0 = C_0(E,h)$ *such that for any* $\delta > 0$ *there exists* $h_0 = h_0(\delta)$ *so that*

$$\sup_{|E-E_0|<\delta} \|\chi R(E,h)\chi\|_{L^2\to L^2} \geq \frac{\log(1/h)}{C_0 h}.$$

for $0 < h < h_0$.

Theorem 1 will follow from the following *Kato's local smoothing estimate.*

Theorem 2. *Suppose that* $E_0 > 0$ *and let* $K(h) \geq 1$ *be a function on* $(0,1)$. *Suppose that for* $|E - E_0| < \delta$ *and* $\chi \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ *we have*

$$\|\chi R(E,h)\chi\|_{L^2\to L^2} \leq \frac{K(h)}{h}.$$

Then for $\varphi \in C_c^\infty((E-\delta, E+\delta), [0,1])$ *and* $u \in L^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}} \|\chi\varphi(P)\exp(-itP/h)u\|_{L^2}^2 \, dt \leq CK(h)\|u\|_{L^2}^2,$$

with a constant $C > 0$ *independent of* h.

In parts I-II, we admit Theorem 2 and show Theorem 1. Part III gives a proof of Theorem 2 for the interested student or if time allows. We follow [3, §7.1].

I – Semiclassical defect measure of a coherent state

The goal of this part is to show

Lemma 3. *Let* $(x_0, \xi_0) \in T^*\mathbb{R}^d$ *and*

$$u_0(x) := (2\pi h)^{-\frac{d}{4}} \exp\Big(\frac{i}{h}\big(\langle x - x_0, \xi_0\rangle + \frac{i}{2}|x - x_0|^2\big)\Big),$$

the coherent state concentrated at (x_0, ξ_0)*. Then, for any* $b \in \mathcal{S}_\gamma(\mathbb{R}^d)$ *with* $0 < \gamma < \frac{1}{2}$,

$$\langle b^w(x, hD)u_0, u_0\rangle = b(x_0, \xi_0) + e(h),$$
$$|e(h)| \leq Ch^{\frac{1}{2}} \max_{|\alpha|=1} \sup_{T^*\mathbb{R}^d} |\partial^\alpha b|,$$

with $C > 0$ *a constant depending only on the dimension* d*.*

Lemma 3 shows that *any* sequence in h from a coherent state concentrated at (x_0, ξ_0) has for defect measure $\delta_{x=x_0,\xi=\xi_0}$, and give a bound on the convergence rate.

1. Show that

$$\langle b^w(x, hD)u_0, u_0\rangle =$$
$$\frac{2^d}{(2\pi h)^{\frac{3d}{2}}} \int\int\int b(z, \xi)e^{\frac{2i}{h}\langle w, \xi - \xi_0\rangle} e^{-\frac{1}{h}(|z - x_0|^2 + |w|^2)} dw\, d\xi\, dz.$$

2. Show that, for z and ξ fixed

$$\int e^{\frac{2i}{h}\langle w, \xi - \xi_0\rangle} e^{-\frac{1}{h}(|z - x_0|^2 + |w|^2)} dw = 2^{-\frac{d}{2}}(2\pi h)^{\frac{d}{2}} e^{-\frac{1}{h}|\xi - \xi_0|^2}.$$

3. Deduce that

$$\langle b^w(x, hD)u_0, u_0\rangle = a_d b(x_0, \xi_0) + e(h),$$

with

$$e(h) := \frac{2^{\frac{d}{2}}}{(2\pi h)^d} \int\int (b(z, \xi) - b(x_0, \xi_0))e^{-\frac{1}{h}(|z - x_0|^2 + |w|^2)}\, dzdw,$$

and a_d a constant depending only on the dimension.

4. Show that $a_d = 1$.

5. Show that

$$|e(h)| \leq h^{\frac{1}{2}} \max_{|\alpha|=1} \sup_{T^*\mathbb{R}^d} |\partial^\alpha b| \frac{2^{\frac{d}{2}}}{(2\pi h)^d} \int\int e^{-\frac{1}{2h}(|z - x_0|^2 + |\xi - \xi_0|^2)}\, dz\, d\xi$$

and conclude.

II – Proof of Theorem 1

The plan is to construct a non-trivial $u_0 \in L^2$ so that, for

$$\varphi \in C_c^\infty((E_0-\delta, E_0+\delta),[0,1]), \quad \varphi(E_0)=1,$$

we have, for some $c>0$

$$\int_{\mathbb{R}} \|\chi\varphi(P)\exp(-itP/h)u_0\|_{L^2}^2 \, dt \geq c\log\frac{1}{h}\|u_0\|_{L^2}^2,$$

and use Theorem 2 to conclude. We recall, from functional calculus for pseudodifferential operators (see for example [2, Chapter 8])

$$\varphi(P(h))\chi^2\varphi(P(h)) = a^w(x,hD), \quad a \in \mathcal{S}(T^*\mathbb{R}^d),$$
$$a(x,\xi) = \chi(x)^2\varphi(p(x,\xi)) + O(h\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty}).$$

Let

$$a_t^w := e^{itP/h}a^w(x,hD)e^{-itP/h}.$$

We will use the following consequence of Egorov's Theorem (see eg [4, Chapter 11]): for $\alpha>0$ sufficiently small and independent of δ, we have, uniformly in $0<t<\alpha\log\frac{1}{h}$

$$a_t \in \mathcal{S}_\gamma(T^*\mathbb{R}^d), \ 0<\gamma<\frac{1}{2},$$
$$a_t - (\exp tH_p)^*a \in h^{2-3\gamma}S_\gamma(T^*\mathbb{R}^d).$$

6. Show that

$$\int_{\mathbb{R}} \|\chi\varphi(P)\exp(-itP/h)u_0\|_{L^2}^2 \, dt \geq \int_0^{\alpha\log\frac{1}{h}} \langle a_t^w(x,hD)u_0, u_0\rangle \, dt.$$

7. Let $(x_0,\xi_0)\in K_{E_0}$. Show that, for any $t\in\mathbb{R}$

$$(\exp tH_p)^*[\chi^2\varphi(p)](x_0,\xi_0) = 1.$$

8. Deduce that

$$a_t(x_0,\xi_0) = 1 + O(h^{\frac{1}{2}}),$$

uniformly for $0<t<\alpha\log\frac{1}{h}$.

9. Conclude by considering the coherent state concentrated at (x_0,ξ_0).

III – Proof of Theorem 2

We let

$$T : u \in L^2(\mathbb{R}^d) \mapsto \chi\varphi(P)e^{-itP/h} \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d),$$

so that by a TT^* argument, it is enough to show that

$$TT^* = O(K(h)) \text{ in } L^2(\mathbb{R} \times \mathbb{R}^d) \to L^2(\mathbb{R} \times \mathbb{R}^d).$$

10. Show that

$$TT^* f = \Big(\chi e^{-i\bullet P/h}\varphi(P)^2\chi\Big) * f,$$

where $*$ denotes the convolution in the time variable.

11. Let the inverse semiclassical Fourier transform

$$\mathcal{F}^{-1}_{t\mapsto\lambda}\psi(\lambda) := \frac{1}{2\pi}\int e^{it\lambda/h}\psi(t)\,dt.$$

We recall the Stone formula: the spectral measure of P writes

$$dE_\lambda(P) = \frac{1}{2\pi i}\big((P-\lambda-i0)^{-1} - (P-\lambda+i0)^{-1}\big)d\lambda.$$

Show that

$$\mathcal{F}^{-1}_{t\mapsto\lambda}\big(e^{-i\bullet P/h}\varphi(P)^2\big) = \frac{1}{2\pi i}\sum_{\pm} \pm(P-\lambda\mp i0)^{-1}\varphi(\lambda)^2.$$

12. Conclude by using the relation between semiclassical Fourier transform and convolution.

Supplementary exercise: complex absorbing potential

Let $V \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ and

$$P := -h^2\Delta + V.$$

Let $a \in C_c^\infty(\mathbb{R}, \mathbb{R}_+)$ be so that P satisfies the following *exterior control condition*:

$$\forall \rho \in T^*\mathbb{R}^d, \quad \pi_x\big(\exp(tH_p)\rho\big) \to \infty \text{ as } t \to -\infty$$
$$\textit{or } \exists t < 0 \text{ s.t. } \pi_x\big(\exp(tH_p)\rho\big) \in \{a > 0\}.$$

By adapting the proof of the non-trapping resolvent estimate seen during the lesson, show that for any $\chi \in C_c^\infty(\mathbb{R}^d)$, there exists $h_0 > 0$ and $C > 0$ so that, for any $f \in L^2(\mathbb{R}^d)$, $E > 0$, and any outgoing solution u to

$$(P - E - ia)u = \chi f,$$

we have, for $0 < h \leq h_0$

$$\|\chi u\|_{L^2} \leq \frac{C}{h} \|f\|_{L^2}.$$

References

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[2] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[3] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[4] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.